A THEOREM ON ENTIRE FUNCTIONS WITH APPLICATIONS TO TOEPLITZ OPERATORS

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1. Introduction. In this note we state and prove a new theorem concerning the valency of entire functions. The initial motivation for Theorem 1 arose in an attempt to answer various questions concerning the commutant of analytic Toeplitz operators. In section **2** we present the proof of Theorem 1 and in section **3** we present its applications to the commutant of analytic Toeplitz operators.

THEOREM 1. *Let γ denote the unit circumference $z = e^{it}$, $0 \leq t \leq 2\pi$. For any F analytic in $\bar{D}$ and $a \notin F(\gamma)$ let $\eta(F(\gamma), a)$ be the winding number of $F(\gamma)$ about a and define*

$$k(F) = \inf \{\eta(F(\gamma), a) : \eta(F(\gamma), a) \neq 0\}.$$

Then for any non-constant entire function f, there exists an entire function h such that $f(z) \equiv h(z^k)$, where $k = k(f)$ and $k(h) = 1$.

Remark. If f is merely analytic in $\bar{D}$, it would be interesting to know whether there exist $\alpha_1, \cdots, \alpha_k \in D$, $k = k(f)$, and a function h analytic in $\bar{D}$ such that $k(h) = 1$ and

$$f(z) = h\left(\left(\frac{z-\alpha_1}{1-\bar{\alpha}_1 z}\right) \cdots \left(\frac{z-\alpha_k}{1-\bar{\alpha}_k z}\right)\right).$$

Let H^2 denote the Hardy space of functions f which are analytic in the open disc $D : \{z : |z| < 1\}$ and are such that the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in L^2-norm for $r < 1$. Let H^∞ denote the space of bounded analytic functions in D. For any $\phi \in H^\infty$ the analytic Toeplitz operator T_ϕ on H^2 is defined by $(T_\phi f)(z) = \phi(z)f(z)$. If ϕ is an entire function, then T_ϕ will be called an entire Toeplitz operator. The operator T_z is called the unilateral shift and has the property that the operators on H^2 which commute with T_z are precisely the analytic Toeplitz operators. For any bounded linear operator A on H^2 it is natural to discuss the commutant $\{A\}' = \{B : AB = BA\}$. The paper [2] contains various results about the commutant of an analytic Toeplitz operator, in particular about those analytic functions ϕ for which $\{T_\phi\}' = \{T_z\}'$.

A function $\chi \in H^\infty$ is called an inner function if the boundary values $\chi(e^{i\theta})$ are of modulus 1 for almost all θ, or equivalently, if T_χ is an isometry. On the other hand $F \in H^\infty$ is called an outer function if T_F has a dense range. Every $\phi \in H^\infty$ has a unique factorization $\phi = \chi F$, where χ is inner and F is outer [4].

The following questions arose in [2].

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