ISOMORPHISMS OF TRANSITIVE OPERATOR ALGEBRAS

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1. Introduction and preliminaries. We propose to record several results on isomorphisms of transitive (irreducible) algebras of Banach-space operators which contain operators of finite rank. Our main results concern sufficient conditions under which such isomorphisms are spatial. The starting point of this study was the observation that closure conditions on these algebras were sometimes irrelevant. Thus we give a strengthening of Rickart's fundamental isomorphism theorem, where he assumed that the algebras were Banach algebras under some norm, (see Theorem 2 below). We also give other related results including applications to self-adjoint and reductive algebras.

In what follows $\mathfrak{X}$ and $\mathcal{Y}$ will be complex Banach spaces; $\mathcal{B}(\mathfrak{X}, \mathcal{Y})$ will denote the set of all bounded linear transformations from $\mathfrak{X}$ into $\mathcal{Y}$; members of $\mathcal{B}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X}, \mathfrak{X})$ will be called *operators* on $\mathfrak{X}$. By a *subspace* of $\mathfrak{X}$ we shall always mean a closed linear manifold in $\mathfrak{X}$. If a linear manifold $\mathfrak{X}_1$ of $\mathfrak{X}$ is invariant under an operator A on $\mathfrak{X}$, then $A \mid \mathfrak{X}_1$ will denote the restriction of A to $\mathfrak{X}_1$; if $\mathcal{S}$ is a set of operators, $\mathcal{S} \mid \mathfrak{X}_1$ will abbreviate the set $\{S \mid \mathfrak{X}_1 : S \in \mathcal{S}\}$.

Let $\mathfrak{A}$ be a subalgebra of $\mathcal{B}(\mathfrak{X})$. (This means that $\mathfrak{A}$ is closed under addition, multiplication and scalar multiplication; $\mathfrak{A}$ does not necessarily contain the identity operator.) We call $\mathfrak{A}$ *transitive* (= topologically irreducible) if the only subspaces invariant under $\mathfrak{A}$ are 0 and $\mathfrak{X}$; we call it *strictly transitive* (= strictly irreducible) if 0 and $\mathfrak{X}$ are the only invariant linear manifolds. If $\mathfrak{A}$ and $\mathfrak{B}$ are subalgebras of $\mathcal{B}(\mathfrak{X})$ and $\mathcal{B}(\mathcal{Y})$ respectively, then by an *isomorphism* ϕ of $\mathfrak{A}$ and $\mathfrak{B}$ we shall mean an algebraic isomorphism (with no topological assumptions); ϕ is called *spatial* if there exists a bicontinuous linear transformation S from $\mathfrak{X}$ onto $\mathcal{Y}$ such that $\phi(A) = SAS^{-1}$ for all A in $\mathfrak{A}$.

Eidelheit [2] proved that every isomorphism of $\mathcal{B}(\mathfrak{X})$ onto $\mathcal{B}(\mathcal{Y})$ is spatial and, therefore, automatically continuous. Rickart [11] generalized this to subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{B}(\mathfrak{X})$ and $\mathcal{B}(\mathcal{Y})$ which are Banach algebras (not necessarily under the operator norm), are strictly transitive, and contain finite-rank operators. This is Rickart's fundamental isomorphism theorem for primitive Banach algebras with minimal ideals. The automatic bicontinuity of isomorphisms for semisimple Banach algebras was proved by Johnson [7].

2. Algebras with no closure conditions. We start with an elementary lemma.

LEMMA 1. *Let $\mathfrak{A}$ be a transitive subalgebra of $\mathcal{B}(\mathfrak{X})$ containing a non-zero finite-rank operator and let ϕ be an isomorphism of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{Y})$. Then there*

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