ON EXPONENTIAL DIVISORS

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Let $\sigma^{(e)}(N)$ denote the sum of the exponential divisors of N, that is, divisors of the form $p_1^{b_1} \cdots p_r^{b_r}$, $b_j \mid a_j$, $j = 1, \cdots, r$, when N has the canonical form $p_1^{a_1} \cdots p_r^{a_r}$, and define $\sigma^{(e)}(1) = 1$. Call N exponentially perfect (or simply e-perfect) if $\sigma^{(e)}(N) = 2N$. We here prove several results concerning e-perfect numbers including the nonexistence of odd e-perfect numbers—thus settling a problem raised earlier. We show that the set $\{\sigma^{(e)}(n)/n\}$ is dense in $[1, \infty)$ and conjecture that the result also holds when $\sigma^{(e)}(n)$ is replaced by any of its iterates. We finally consider the structure of the semigroup of artihmetic functions under exponential convolution.

1. Introduction. By an "exponential divisor" (or e-divisor) of a positive integer $N > 1$ with canonical form

$$N = p_1^{a_1} \cdots p_r^{a_r} \tag{1.1}$$

we mean a divisor d of N of the form

$$d = p_1^{b_1} \cdots p_r^{b_r}, \qquad b_j \mid a_j, \qquad j = 1, \cdots, r.$$

The number and sum of such divisors of N are denoted respectively by $\tau^{(e)}(N)$ and $\sigma^{(e)}(N)$. By convention, 1 is an exponential divisor of itself so that $\tau^{(e)}(1) = \sigma^{(e)}(1) = 1$.

The definition and notation used here are the same as in [4] where these functions are considered in some detail.

It is evident that $\tau^{(e)}(N)$ and $\sigma^{(e)}(N)$ are multiplicative functions, and hence

$$\tau^{(e)}(N) = \tau(a_1) \cdots \tau(a_r),$$

$$\sigma^{(e)}(N) = \prod_{j=1}^{r} \sigma^{(e)}(p_j^{a_j}) = \prod_{j=1}^{r} \Big(\sum_{b_j \mid a_j} p_j^{b_j}\Big)$$

where $\tau(a)$ denotes, as usual, the number of divisors of a.

In Section 2 we obtain some results concerning exponentially perfect (or briefly, e-perfect) numbers, that is, integers N for which $\sigma^{(e)}(N) = 2N$, and we settle a question raised in [4] (see also [5]) by proving that there are no odd e-perfect numbers. Actually, we prove in Theorem 2.2 in the sequel a more general result.

In Section 3 we show that every number greater than or equal to 1 is a limit point of the set

Received July 17, 1973. Revisions received October 12, 1973. The first author was supported in part by NSF grant GP-28696. The second author was supported in part by NRC grant A-3103.