ORIENTABLE IMBEDDINGS OF CAYLEY GRAPHS

ARTHUR T. WHITE

1. Introduction. A graph G is said to be *imbedded* in a closed orientable 2-manifold M if the geometric realization of G as a finite 1-complex is homeomorphic to a subspace of M. The components of the complement of G in M are called *regions*. A region which is homeomorphic to an open disk is called a *2-cell*; if every region is a 2-cell, the imbedding is said to be a *2-cell imbedding*. If M has genus k, we write $M = S_k$. The *genus* of a graph G, denoted by $\gamma(G)$, is the smallest k $(k \geq 0)$ for which G has an imbedding in S_k. It is well-known (see, for example, [19, p. 198]) that, if G is connected, any imbedding of G on $S_{\gamma(G)}$ (i.e., a *minimal imbedding*) must be a 2-cell imbedding. Clearly, if G is disconnected, no imbedding of G will be 2-cell.

It is without loss of generality that we restrict our attention to connected graphs when studying the genus parameter, since Battle, Harary, Kodama, and Youngs [2] have shown that the genus of a disconnected graph is the sum of the genera of its components. The *maximum genus* of a graph G, denoted by $\gamma_M(G)$, is the largest k $(k \geq 0)$ for which G has a 2-cell imbedding in S_k. Duke [6] has shown that, if $\gamma(G) \leq k \leq \gamma_M(G)$, then G has a 2-cell imbedding in S_k. The maximum genus parameter has been studied extensively by Nordhaus, Stewart, Ringeisen, and the author (see [22], [23], and [25]).

There are relatively few families of graphs for which the genus is known. In 1955, Ringel [27] showed that the genus of the n-cube Q_n is given by: $\gamma(Q_n) = 1 + 2^{n-3}(n - 4)$, for $n \geq 2$. This formula was established independently by Beineke and Harary [3] in 1965. In 1963 Auslander, Brown, and Youngs [1] produced a family of graphs G_n for which $\gamma(G_n) = n$. In 1965 Ringel [28] found the genus of all complete bipartite graphs: $\gamma(K_{m,n}) = \{((m - 2)(n - 2))/4\}$, for $m, n \geq 2$. Ringel and Youngs [30] settled the Heawood map-coloring conjecture in the affirmative in 1968, by showing that the complete graph has genus given by: $\gamma(K_n) = \{((n - 3)(n - 4))/12\}$, for $n \geq 3$. In 1969 Ringel and Youngs [31] (and independently the author [33]) found the genus of all regular complete tripartite graphs: $\gamma(K_{n,n,n}) = ((n - 1)(n - 2))/2$. Also in 1969, Jacques [18] found the genus of certain Cayley graphs π_n for the symmetric group S_n. During the period 1969–1972, the author found a number of other genus formulae; see [32], [33], [34], [35], and [36]. Recently Himelwright [17] has shown that $\gamma(K_{4,4} \times Q_n) = n2^n + 1$, for $n \geq 0$, where "$\times$" denotes the cartesian product.

The maximum genus has been computed for some of the graphs for which

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