THE TOPOLOGICAL SPACE OF ORDERINGS OF A RATIONAL FUNCTION FIELD

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Given a formally real field F, its set of orderings $X(F)$ can be topologized to give a Boolean space (i.e., compact, Hausdorff and totally disconnected). This topological space arises naturally from looking at the Witt ring of the field, and has been studied by Knebusch, Rosenberg and Ware in the more general case where F is a semilocal ring and one considers signatures rather than orderings [7]. In this paper, the space of orderings $X(F)$ is investigated in the case in which F is a rational function field. It is proved that in this case $X(F)$ is always perfect, which leads to a characterization of the rational function fields for which $X(F)$ is homeomorphic to the Cantor set. An elementary topological proof is given for the rational function field case of a theorem discovered independently by Elman, Lam and Prestel [5]; their proof involves a long analysis of the behavior of quadratic forms. It is proved that the rational function field $F(x)$ satisfies the strong approximation property iff F is hereditarily euclidean, a class of fields studied in depth by Prestel and Ziegler [11].

1. Notation. All fields in this paper will be formally real; i.e., they can be ordered in at least one way. If F is an ordered field, $\tilde{F}$ will denote the real closure of F with respect to its ordering, and $\dot{F}$ will denote the multiplicative group of nonzero elements of F. The field $F(x)$ will always be the rational function field in one indeterminate x over F.

The topological space of orderings of F will be denoted by $X(F)$, where the topology is generated by the subbasis consisting of all sets of the form $W_F(a) = \{< \in X(F) \mid a < 0\}$ for $a \in \dot{F}$ (cf. [9] or [8]). The collection of all such subsets of $X(F)$ will be denoted by $\mathcal{H}(F)$ and will be called the Harrison subbasis. When the field is clear, we shall write $W(a)$ and $\mathcal{H}$ for $W_F(a)$ and $\mathcal{H}(F)$. Since the complement of $W(a)$ is $W(-a)$, these sets are all clopen (both closed and open). Also, $W(ab) = W(a) + W(b)$, where $+$ denotes symmetric difference, since ab is negative iff a or b is negative but not both.

2. General Results. We say that F satisfies SAP (strong approximation property, originally defined in [8]) if given any two disjoint closed sets A, $B \subseteq X(F)$, there exists an element $a \in \dot{F}$ such that $B \subseteq W(a)$ and $A \subseteq W(-a)$. The importance of this property is that it leads to an explicit computation of the reduced Witt ring of F [7, Cor. 3.21]. It is generally easier to use the following equivalent forms of SAP:

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