ENDOMORPHISM RINGS OF FREE MODULES OVER PRINCIPAL IDEAL DOMAINS

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One of the fundamental problems on endomorphism rings is to find necessary and sufficient conditions that an abstract ring be isomorphic to the endomorphism ring of a given type of abelian group (see chapter XV in [3]. The prototype of theorems along this line is Wolfson's beautiful characterization of the ring of all linear transformations of a vector space over a division ring [13]. We have solved this problem for various types of abelian groups without using matrix representations for endomorphisms (see [7]–[12]).

The purpose of the present paper is to determine criteria under which an abstract ring E is the ring $E(F)$ of all R-endomorphisms of a free module F over a (not necessarily commutative) principal ideal domain R. It is to be expected that the idempotent elements in $E(F)$ play a key role in this characterization. Our methods partly originate from [9] and [12]. The conditions we impose on E in Theorem 3.1 are such that E is a dense subring in the finite topology of $E(F)$. This topology is the usual finite topology on transformation groups. Completeness of E in this topology then blows up E to the full ring $E(F)$. Wolfson did not use any topological arguments in his characterization theorem. Therefore we do not obtain his result as a corollary.

Throughout this paper R denotes a (not necessarily commutative) principal ideal domain. Unless otherwise indicated, all modules will be unitary left modules. F will always be a free R-module. Any two bases of F have the same cardinal ([4], Theorem 17, p. 43). We call it the rank of F and denote it by $r(F)$. All homomorphisms of modules will be written on the right. $\bigoplus$ is our symbol for direct sum, and $\langle S\rangle$ stands for the submodule generated by the subset S of F.

1. The Finite Endomorphisms. An R-endomorphism α of F is called *finite* if the submodule $F\alpha$ of F is finitely generated. Let $E_0(F)$ denote the set of all finite R-endomorphisms of F. It is immediate that $E_0(F)$ is a two-sided ideal of $E(F)$. Our first objective is to characterize $E_0(F)$ inside $E(F)$ in purely ring-theoretical terms.

(1.1) LEMMA. *Let M be a submodule of F such that F/M is finitely generated. Then there exists a decomposition $F = F_1 \bigoplus F_2$ of F with $r(F_1) < \infty$ and $F_2 \subseteq M$.*

Proof. We have finitely many elements $x_1, \cdots, x_n$ in F with $F = \langle M, x_1, \cdots, x_n\rangle$. There exists a decomposition $F = F_1 \bigoplus G$ of F with $r(F_1) < \infty$ and $x_i \in F_1$ for $i = 1, \cdots, n$. Here G is free, say, $G = \bigoplus_{i\in J} \langle g_i\rangle$. Since

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