LIMITS OF UNITARILY EQUIVALENT NORMAL OPERATORS

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1. Introduction. Two bounded operators A and B on a Hilbert space are abstractly identical if they are unitarily equivalent, i.e., if there is a unitary operator U such that $B = UAU^*$.

Our interest is in examining the geometric and topological structure of the equivalence classes in $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$, determined by unitary equivalence as an equivalence relation. Some such equivalence classes are totally trivial. For example, any scalar multiple of the identity lies in an equivalence class by itself. Other equivalence classes are much more complex, and what information is known is of a rather specialized nature. Some are closed, but not all. The easiest general statement that can be made is that each equivalence class is arcwise connected. This is a quite simple consequence of the fact that the unitaries form an arcwise connected set in $\mathcal{B}(\mathcal{H})$. (See Halmos, [1, p. 66]).

The question that motivated this research is the following: Does each of the equivalence classes in $\mathcal{B}(\mathcal{H})$ have arcwise connected closure? This question remains open. The most significant operators for which we have been able to answer the question are the normal ones. An equivalence class made up of normal operators necessarily has an arcwise connected closure. (See Theorem 2.) Finally we look at an example that indicates an essential difference in the normal and non-normal cases.

After this research was completed Halmos' paper [2] on limits of shifts appeared. His theorem offers a complete description of the closure of any equivalence class of unilateral or bilateral shifts. Theorem 1 is very closely related to Halmos' Lemma 4 and contains the "bilateral shift" part of his theorem.

II. Normal operators. Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. For each fixed operator A we designate by $\mathcal{U}(A)$ the set of all operators in $\mathcal{B}(\mathcal{H})$ that are unitarily equivalent to A, and by $\mathcal{U}(A)^-$ the closure of $\mathcal{U}(A)$ in the uniform operator topology. The spectrum of A is denoted $\sigma(A)$. It is elementary to check that for any pair of operators A and B, $\mathcal{U}(A)^-$ and $\mathcal{U}(B)^-$ are either disjoint or equal, i.e., no partial overlapping is possible.

The first part of the following lemma is elementary and direct. The second part is a simple consequence of the Weierstrass approximation theorem and the spectral mapping theorem and was noted by Halmos [2].

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