EVERY PLANAR GRAPH HAS AN ACYCLIC 8-COLORING

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An acyclic k-coloring of a graph G is an assignment of k or fewer colors to $V(G)$ such that each element of $V(G)$ has exactly one color, adjacent vertices are colored differently, and any set of vertices with two colors induces an acyclic graph. The minimum k for which G has an acyclic k-coloring is denoted by $a(G)$ and is called the acyclic chromatic number of G. Grünbaum [3] introduced the concept of acyclic coloring, which is related to point-arboricity. (See, for example, [1] or [4].) Although Grünbaum conjectured that for any planar graph G, $a(G) \leq 5$, he was only able to prove that $a(G) \leq 9$. In this paper we show for any planar graph $a(G) \leq 8$.

The degree of a vertex in graph G is denoted $\deg_G v$. The set of vertices of degree i is denoted F_i and has cardinality f_i. We begin with the following remarks and lemma.

Remark 1. Any maximal plane graph with at least four vertices has minimum degree of at least three.

Remark 2. Let v be a vertex of a maximal plane graph with at least four vertices. If $u_1, u_2, \cdots, u_k$ are the vertices adjacent to v taken consecutively in a clockwise fashion, then $u_1, \cdots, u_k, u_1$ is a cycle of G.

LEMMA. *If G is a maximal plane graph with minimum degree four or five, then at least one of the following holds.*

(i) *There exists $v \in F_5$ which is adjacent to two vertices of degree less than seven.*

(ii) *There exists $v \in F_5$ which is adjacent to an element of F_5 and an element of F_7.*

(iii) *There exists $v \in F_4$ which is adjacent to a vertex of degree less than eight.*

Proof. Let $v_1, \cdots, v_k$ be the elements of F_5 which are adjacent with exactly one element of F_5, and let $v_{k+1}, \cdots, v_n$, $n = f_5$, be the remaining elements of F_5. Also denote the elements of F_4 by $u_1, \cdots, u_{f_4}$, and assume that the lemma is false. This implies that each v_i, $1 \leq i \leq k$, and each u_j, $1 \leq j \leq f_4$, is adjacent with exactly four vertices of degree at least eight. Thus the elements of F_7 are not adjacent with any u_j or any v_i, $1 \leq i \leq k$. For $k < i \leq n$ each v_i is adjacent to four vertices of degree at least seven and the remaining adjacency has degree at least six. This implies that the only elements of $F_4 \cup F_5$ adjacent to elements of F_7 are v_i, $k < i \leq n$. No two of these v_i are adjacent. From Remark 2 it follows that each element of F_7 is adjacent to at most three elements of $F_4 \cup F_5$. Hence there are at most $3f_7$ edges joining F_7 to $F_4 \cup F_5$.

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