DENTABILITY AND THE RADON-NIKODÝM PROPERTY

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Let X be any Banach space which contains a nondentable bounded subset K in the sense of M. A. Rieffel. We construct an X-valued measure ν of bounded variation which is absolutely continuous with respect to a bounded scalar measure λ but which does not have a derivative with respect to λ. This improves upon a construction by H. Maynard who assumed the set K was not σ-dentable, and, in conjunction with a result of Rieffel, this shows that a Banach space has the Radon–Nikodým property if and only if every bounded subset is dentable.

1. Introduction. If K is a bounded set in a (real) Banach space X (with norm $\|\cdot\|$), the σ-convex hull of K is the set $\sigma\text{-co}\,(K) = \{\sum_{i=1}^{\infty} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{\infty} \alpha_i = 1, x_i \in K\}$. Trivially, $\text{co}\,(K) \subset \sigma\text{-co}\,(K) \subset \overline{\text{co}}\,(K)$, where $\text{co}\,(K)$ and $\overline{\text{co}}\,(K)$ denote the convex hull and the closed convex hull of K respectively. The set K is said to be *dentable* (respectively *σ-dentable*) provided that for every $\epsilon > 0$ there exists a point x in K such that $x \notin \overline{\text{co}}\,(K \backslash B_\epsilon(x))$ (respectively $x \notin \sigma\text{-co}\,(K \backslash B_\epsilon(x))$), where $B_\epsilon(x)$ is the open ball of radius ϵ centered at x.

Throughout, Σ will always denote a σ-algebra of subsets of some set Ω; $\text{ca}^+(\Sigma)$ denotes the set of all nonnegative, bounded, countably additive measures on Σ; if λ is in $\text{ca}^+(\Sigma)$, then $L^1(\lambda, X)$ denotes the space of all Bochner-λ-integrable functions $g : \Omega \to X$. The space X is said to have the *Radon–Nikodým property* (RNP) if for every Σ, every λ in $\text{ca}^+(\Sigma)$, and every X-valued measure ν which is of bounded variation and absolutely continuous with respect to λ, there exists some g in $L^1(\lambda, X)$ such that $\nu(E) = \int_E g \, d\lambda$ for all E in Σ.

M. A. Rieffel [4] introduced the notion of dentability and proved that X has the RNP whenever every bounded subset of X is dentable. H. Maynard [3], introducing the weaker notion of σ-dentability, observed that Rieffel's proof held if dentability was replaced by σ-dentability, and he proved the corresponding converse. That is, Maynard proved that (i) implies (iii) in the following theorem, Rieffel's result being that (iii) implies (i).

THEOREM. *For a Banach space X the following three statements are equivalent.*
(i) *X has the* RNP.
(ii) *Every bounded subset of X is dentable.*
(iii) *Every bounded subset of X is σ-dentable.*

Maynard's proof involves a rather complicated construction. In this note we prove directly that (i) implies (ii) and hence complete the proof of the equivalence of all three statements. (Note that (ii) implies (iii) by definition.)

Received August 7, 1973.