PARTIALLY ORDERED SETS ASSOCIATED WITH FIBONACCI REPRESENTATIONS

L. CARLITZ AND RICHARD SCOVILLE

1. Introduction. We define the Fibonacci numbers $\{F_n\}$ as usual by means of

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1}, \qquad n \geq 1. \tag{1.1}$$

We let $R_k(N)$ be the number of ways the integer N can be written

$$N = \epsilon_2 F_2 + \epsilon_3 F_3 + \cdots, \qquad \epsilon_i = 0, 1, i = 2, 3, \cdots, \tag{1.2}$$

with

$$\epsilon_2 + \epsilon_3 + \cdots = k. \tag{1.3}$$

At least one way is always possible, namely, the Zeckendorf or *canonical* representation [1] of N,

$$N = \epsilon_2 F_2 + \cdots + \epsilon_n F_n, \qquad \epsilon_i \epsilon_{i+1} \neq 1, \qquad i = 2, 3, \cdots. \tag{1.4}$$

The purpose of this paper is to continue the study of the numbers $R_k(N)$ and the related polynomials

$$R(N, t) = R(N) = \sum t^k R_k(N) \tag{1.5}$$

along the lines of [1], [2] and [3].

In Section 2 we show that the ways in which N can be written in the form (1.2) give rise to a partially ordered set $G(N)$ or, more precisely, to four partially ordered sets

$$\tilde{G}(N) = \begin{bmatrix} G_1(N) & G_1'(N) \\ G_0(N) & G_0'(N) \end{bmatrix}. \tag{1.6}$$

We introduce a composition of two numbers $N \circ M$ and an associated composition $\tilde{G}(N) \circ \tilde{G}(M)$ and show that

$$\tilde{G}(N \circ M) = \tilde{G}(N) \circ \tilde{G}(M). \tag{1.7}$$

In Section 3 we introduce certain 2×2 matrices $Q(N, t)$ each entry of which is a polynomial in t. We show that

$$Q(N \circ M) = Q(M) \cdot Q(N) \tag{1.8}$$

if N and M are suitably restricted. The composition $N \circ M$ is defined using the Zeckendorf representations (1.4) of N and M. Suppose that

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