ALGEBRAS CONTAINING UNILATERAL SHIFTS OR FINITE-RANK OPERATORS

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1. Introduction. An algebra $\mathfrak{A}$ of bounded operators on a Hilbert space is *reductive* if it is weakly closed, contains the identity operator, and has the property that every subspace invariant under (all the operators in) $\mathfrak{A}$ reduces (all the operators in) $\mathfrak{A}$. Von Neumann algebras are obviously reductive; the *reductive algebra problem* (raised in [13]) is the question: Is every reductive algebra a von Neumann algebra? Equivalently, if $\mathfrak{A}$ and $\mathfrak{A}^*$ have the same invariant subspaces, must $\mathfrak{A} = \mathfrak{A}^*$?

There are a number of known results on transitive algebras (operator algebras with no nontrivial invariant subspaces) [1], [2], [6], [10], [11] and [12]. Most of these results have recently been generalized to partial solutions of the reductive algebra problem [4], [8], [9] and [13] ([14] contains discussions of all of those theorems). We prove another such generalization in the present note, i.e., we generalize the result of [11] to prove that a reductive algebra containing a unilateral shift of finite multiplicity is a von Neumann algebra. We prove this theorem by using the basic lemma of [11] and the techniques introduced in [4]. We must modify the techniques of [4] to deal with unbounded operators, and this requires certain lemmas about closed operators which commute with unilateral shifts. These lemmas might also be useful in other contexts.

We also prove that a reductive algebra which is generated by finite-rank operators is self-adjoint; this was conjectured during a conversation with Abie Feintuch.

With these results all the known theorems about transitive algebras have now been generalized to the reductive case.

2. Preliminaries on reductive algebras. We recall certain basic facts useful in the study of reductive algebras. As usual, we define $A^{(n)}$ for $A \in \mathcal{B}(\mathcal{H})$ and n a positive integer as the direct sum of n copies of A acting on the direct sum of n copies of $\mathcal{H}$. If $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathfrak{A}^{(n)} = \{A^{(n)} : A \in \mathfrak{A}\}$. The following is a standard lemma whose proof is very easy, [13; Lemma 2].

LEMMA 1. *If $\mathfrak{A}^{(n)}$ is a reductive algebra for every positive integer n, then $\mathfrak{A}$ is self-adjoint.*

The next lemma is a slight generalization of [4; Lemma 3].

LEMMA 2. *If $\mathfrak{A}$ is a reductive algebra and T is a closed linear transformation commuting with $\mathfrak{A}$ such that $T^2 = 0$, i.e., the range of T is included in the direct*

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