A CONTINUED FRACTION OF CARLITZ

M. D. HIRSCHHORN

Let

$$(x)_n = 1 - x^n, \quad (x)_n\,! = (x)_1(x)_2 \cdots (x)_n .$$

The main result of this note is that

$$1 + ax + \frac{abx^2}{1 + ax^3 +} \cdots \frac{abx^{2n}}{1 + ax^{2n+1}} = \frac{P_n(a, b, x)}{Q_n(a, b, x)},$$

where

$$\begin{aligned} P_n(a, b, x) &= \sum_{r=0}^{n+1} a^r x^{r^2} \sum_{s=0}^{\min(r, n+1-r)} b^s x^{s^2} \frac{(x^2)_{n+1-s}!}{(x^2)_s!\,(x^2)_{r-s}!\,(x^2)_{n+1-r-s}!}, \\ Q_n(a, b, x) &= \sum_{r=0}^{n} a^r x^{r^2+2r} \sum_{s=0}^{\min(r, n-r)} b^s x^{s^2} \frac{(x^2)_{n-s}!}{(x^2)_s!\,(x^2)_{r-s}!\,(x^2)_{n-r-s}!}. \end{aligned} \tag{1}$$

If we take $|x| < 1$, let $n \to \infty$, and use the well-known identity [2; Theorem 348], [3; Art. 246]

$$(1 + cx)(1 + cx^2) \cdots (1 + cx^r) = \sum_{s=0}^{r} c^s x^{\frac{1}{2}s(s+1)} \frac{(x)_r!}{(x)_s!\,(x)_{r-s}!}$$

with x^2 for x and b/x for c, then we obtain the following result of Carlitz [1; (10)], namely,

$$\begin{aligned} 1 + ax + \frac{abx^2}{1 + ax^3 +} \frac{abx^4}{1 + ax^5 +} \cdots \\ &= \sum_{r=0}^{\infty} a^r x^{r^2} \sum_{s=0}^{r} b^s \frac{x^{s^2}}{(x^2)_s!\,(x^2)_{r-s}!} \Big/ \sum_{r=0}^{\infty} a^r x^{r^2+2r} \sum_{s=0}^{r} b^s \frac{x^{s^2}}{(x^2)_s!\,(x^2)_{r-s}!} \\ &= \sum_{r=0}^{\infty} a^r x^{r^2} \frac{(1 + bx) \cdots (1 + bx^{2r-1})}{(x^2)_r!} \\ &\quad \Big/ \sum_{r=0}^{\infty} a^r x^{r^2+2r} \frac{(1 + bx) \cdots (1 + bx^{2r-1})}{(x^2)_r!}. \end{aligned}$$

We shall require the following result on partitions [3; Art. 241], namely,

$$\sum_k p(k, r, n) x^k = \frac{(x)_{r+n}!}{(x)_r!\,(x)_n!},$$

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