APPROXIMATION IN THE MEAN BY SOLUTIONS OF ELLIPTIC EQUATIONS

LARS INGE HEDBERG

Let E be a compact set in $\mathbf{R}^d$. In the report [12] we gave conditions on E which insured that functions harmonic in neighborhoods of E are dense in the subspace of $L^p(E)$, $1 \leq p < \infty$, which consists of functions harmonic in the interior of E. J. Polking in [21] independently investigated the same problem for more general classes of elliptic equations and obtained similar results. It turns out that by combining the methods of [12] and [21] one can improve these results, and that is the purpose of this paper.

The corresponding approximation problem for analytic functions of one complex variable has been studied by several authors. We refer to [5], [10], [11], [13], and the references given there.

We limit the discussion to the equation

$$\Delta^{m/2} u = 0, \tag{1}$$

where $m \geq 2$ is an even integer. The results hold, however, for more general classes of equations. See Polking [21].

The interior of a set E will be denoted by E^0. The set of functions u which are solutions of (1) in some neighborhood of E is denoted by $h_m(E)$, and the subspace of $L^p(E)$ which consists of functions satisfying (1) in E^0 is denoted by $h_m^p(E)$. Throughout the paper q will mean $p/(p-1)$. We write

$$B_x(\delta) = \{y \in \mathbf{R}^d : |y - x| < \delta\}.$$

Various constants are denoted by A.

The Sobolev space $W_m^q(\mathbf{R}^d) = W_m^q$, $1 < q < \infty$, consists of all functions in $L^q(\mathbf{R}^d)$ whose distribution derivatives of all orders less than or equal to m are in L^q. Partial derivatives of a function v are denoted by $D^\alpha v$, where $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. It is well known that $||v||_{m,q} = \{\int_{\mathbf{R}^d} \sum_{|\alpha|=m} |D^\alpha v|^q \, dx\}^{1/q}$ is a norm on W_m^q. See, e.g., Stein [22; Chapter V, §2]. We denote the set of functions in W_m^q which vanish a.e. outside a compact set E by $(W_m^q)_E$.

The following fact is well known. A proof is given in [21].

Lemma 1. *$h_m(E)$ is dense in $h_m^p(E)$, $1 < p < \infty$, if and only if $C_0^\infty(E^0)$ is dense in $(W_m^q)_E$. If $E^0 = \emptyset$, $h_m(E)$ is dense in $h_m^p(E)$, $1 < p < \infty$, if and only if $(W_m^q)_E = \{0\}$.*

Thus, from now on we will consider only this dual approximation problem

Received April 6, 1972.