ON A CONJECTURE OF MAHLER IN THE GEOMETRY OF NUMBERS

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1. Let K be a closed convex body in euclidean n-space R_n that is symmetric in the origin 0. For a real and positive number t denote by $K(t)$ that part of K which satisfies $|x_n| \leq t$, where x_n is the n-th coordinate of a fixed cartesian coordinate system. If $V(K)$ denotes the volume of K, then it follows from the Brunn-Minkowski theorem that $V(K(t))/t$ is a monotone decreasing function of t. Mahler [1] has conjectured that the same holds true for the critical determinant of K. This number is defined as the infimum of the determinants of all those lattices which do not contain an inner point of K apart from 0. Mahler proved his conjecture when $n = 2$ and our objective here is to prove the same result for $n = 3$.

Mahler's conjecture appears related to another conjecture in the geometry of numbers, namely, if L is a lattice of determinant $d(L)$ and if $u_1(L), \cdots, u_n(L)$ are the successive minima of L with respect to K, then

(i) $$u_1(L)u_2(L) \cdots u_n(L)\, \Delta(K) \leq d(L),$$

where $\Delta(K)$ denotes the critical determinant of K. This conjecture is well known to be true for $n = 2$ and a proof has been given for $n = 3$ [3]. The latter proof will form the crucial step in our proof.

An interesting example is afforded by the sawn-off cube in R_3; for according to Whitworth [2], the region $K(t)$ defined by the inequalities $|x_i| \leq 1$ for $i = 1, 2, 3$ and $|x_1 + x_2 + x_3| \leq t$, $0 < t \leq \frac{1}{2}$, has $V(K(t))/t = (9 - t^2)/12$ and $\Delta(K(t))/t = \frac{3}{4}$. Thus the two functions are not necessarily constant simultaneously.

2. It is well known that any convex body symmetric in 0 can be approximated arbitrarily closely by strictly convex bodies symmetric in 0 and, further, that the critical determinant of a body varies continuously with the body . Hence it is clear that it suffices to prove the theorem only for strictly convex bodies K. Thus from now on we assume that K is strictly convex.

It is also well known and easily proved that $\Delta(K(t))$ is a continuous function of t. Hence, in order to prove the theorem it suffices to show that for any positive real t and for all sufficiently small positive ϵ

(ii) $$\Delta(K(t))/t \geq \Delta(K(t + \epsilon))/(t + \epsilon).$$

From now on t will be a fixed positive real number. Let L be a critical lattice of $K(t)$, i.e., a lattice of determinant $d(L) = \Delta(K(t))$ which contains no inner

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