PRIMITIVE NUMBERS FOR A CLASS OF MULTIPLICATIVE FUNCTIONS

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1. Introduction. Letting

$$\Sigma(n) = \frac{\sigma(n)}{n} = \frac{1}{n}\sum_{d|n} d,$$

an integer n is called abundant if $\Sigma(n) \geq 2$. The integer n is called primitive or primitive abundant if n is abundant but no proper divisor of n is abundant. Letting $\nu(n) = \sum_{p|n} 1$ be the number of distinct prime divisors of n, Dickson has shown [1] the following.

THEOREM 1. *For each integer k there are at most finitely many odd primitive numbers n with $\nu(n) = k$.*

In [2] H. N. Shapiro enlarges upon the notions of abundant and primitive numbers. Namely, for $\eta > 0$ an integer n is called η-abundant if $\Sigma(n) \geq \eta$, and n is called η-primitive if it is η-abundant but no proper divisor is η-abundant. In this setting an extension of Theorem 1, as found in [2], provides the next theorem.

THEOREM 2. *For η rational there are at most finitely many η-primitive numbers n with $\nu(n) = k$ and such that n is relatively prime to the numerator of η (in reduced form). For η irrational there are altogether only finitely many η-primitive numbers with $\nu(n) = k$.*

In proving Theorem 2 it is shown in [2] that a necessary condition for the existence of infinitely many η-primitive numbers with $\nu(n) = k$ is that there exist integers a and b such that

$$(1) \qquad \begin{cases} \eta = \dfrac{\sigma(a)}{a}\cdot\dfrac{b}{\phi(b)} \quad \text{with} \quad (a, b) = 1 \quad \text{and} \quad b > 1 \\ \nu(a) + \nu(b) < k \end{cases}$$

where ϕ is the Euler function.

In a more recent paper by Shapiro [3] it is shown that (1) is also sufficient, which yields the following theorem.

THEOREM 3. *A necessary and sufficient condition that there be infinitely many η-primitive numbers n with $\nu(n) = k$ is that there exist integers a and b satisfying* (1).

Here we make use of the methods of [3] to produce a generalization of Theorem

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