# A THEORY OF INTEGRATION

B. S. THOMSON

This paper presents an exposition of the ideas fundamental to a theory of integration which has been investigated by R. Henstock [3], [4] and [5] and later by E. J. McShane [6]. The emphasis of these authors has been on a Riemann-type definition of an integral which possesses Lebesgue-type limit theorems and in particular on the problems of defining such an integral for vector-valued functions.

There is an underlying simplicity in this area which is obscured by a Riemann-oriented approach. We present here, in what seems to be the simplest kind of setting, the basic ideas of that part of the theory which interacts with the measure-theoretic tradition. The generalized versions of Henstock and McShane can then be considered to expand this setting. The paper concludes with a brief application of the theory to the familiar problem of integration in locally compact spaces.

**1. Basic theory.** Throughout the paper $\mathbf{N}$ will denote the natural numbers, $\mathbf{R}$ the real numbers, $\mathbf{R}_+$ the nonnegative real numbers, $\bar{\mathbf{R}}_+$ the extended nonnegative real numbers and $\bar{\mathbf{R}}_+^T$ the collection of all functions defined on a set $T$ with values in $\bar{\mathbf{R}}_+$.

Let $T$ be a set and $\mathbf{I}$ a collection of pairs $(I, x)$ where $x \in T$ and $I \subseteq T$. A subset $\mathbf{D}$ of $\mathbf{I}$ is said to be *disjointed* if the corresponding collection $\{I: (I, x) \in \mathbf{D}\}$ is disjointed; a finite disjointed subset $\mathbf{D}$ of $\mathbf{I}$ is called a *division* and we write $\sigma(\mathbf{D}) = \bigcup \{I : (I, x) \in \mathbf{D}\}$ and call such sets $\sigma(\mathbf{D})$ *elementary sets*. The family of all elementary sets is denoted $\mathfrak{E}$.

If $X \subseteq T$, $\mathbf{S} \subseteq \mathbf{I}$ and $\mathfrak{A}$ is a family of subsets of $\mathbf{I}$, we define the following special sets.

$$\mathbf{S}(X) = \{(I, x) \in \mathbf{S} : I \subseteq X\}$$
$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S} : x \in X\}$$
$$\mathfrak{A}(X) = \{\mathbf{S}(X) : \mathbf{S} \in \mathfrak{A}\}$$
$$\mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\}$$

DEFINITION 1. The ordered triple $(T, \mathfrak{A}, \mathbf{I})$ is said to be a *division system* if $\mathfrak{A}$ is a collection of subsets of $\mathbf{I}$ directed downwards by set inclusion, i.e., if $\mathbf{S}_1, \mathbf{S}_2 \in \mathfrak{A}$, then there is an $\mathbf{S} \in \mathfrak{A}$ such that $\mathbf{S} \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$.

DEFINITION 2. A division system $(T, \mathfrak{A}, \mathbf{I})$ is said to be *fully decomposable* (respectively *decomposable*) if for every family (respectively countable family)

Received March 10, 1972. Revision received June 8, 1972.