CONTINUITY OF POSITIVE AND MULTIPLICATIVE FUNCTIONALS

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Our purpose here is to prove a theorem whose consequences show that the validity of certain well-known theorems about Banach and Fréchet locally multiplicatively-convex algebras depends only on the fact that the underlying topology of the algebra is complete and metrizable.

THEOREM 1. *Let H be a complete metrizable abelian topological group whose composition is denoted additively, and let s be a continuous function from H into H such that $s(0) = 0$. If f is a homomorphism from H into the additive group of the topological field* $\mathbf{R}$ *of real numbers such that for some $K > 0$, $f(x)^2 \leq Kf(s(x))$ for all $x \in H$, then f is continuous.*

Proof. Suppose that f is not continuous. Then f is not continuous at zero, so there exist $e > 0$ and a sequence $(a_k)_{k \geq 0}$ in H such that $\lim a_k = 0$ but $f(a_k) \geq e$ for all $k \geq 0$. Let m be a natural number such that $m \geq K/e$ and let $g: H^2 \rightarrow H$ be defined by $g(x_1, x_2) = x_1 + m.s(x_2)$. We define $g_k : H^{k+1} \rightarrow H$ recursively as follows: g_0 is the identity function on H and

$$g_k(x_1, \cdots, x_{k+1}) = g(x_1, g_{k-1}(x_2, \cdots, x_{k+1})) \tag{1}$$

for all $k \geq 1$. Clearly each g_k is continuous. An inductive argument establishes that

$$g_k(x_1, \cdots, x_k, 0) = g_{k-1}(x_1, \cdots, x_k) \tag{2}$$

for all $k \geq 1$. Let $(V_n)_{n \geq 1}$ be a fundamental system of neighborhoods of zero in H satisfying

$$V_{n+1} + V_{n+1} \subseteq V_n \tag{3}$$

for all $n \geq 1$. We shall define recursively a subsequence $(b_n)_{n \geq 0}$ of $(a_k)_{k \geq 0}$ as follows. Let $b_0 = a_0$; if $b_0, \cdots, b_n$ are defined and if $b_n = a_{k_n}$, let $b_{n+1} = a_r$ where r is the smallest $t > k_n$ such that

$$g_{n-k+1}(b_k, \cdots, b_n, a_t) - g_{n-k+1}(b_k, \cdots, b_n, 0) \in V_{n+1},$$

$0 \leq k \leq n$; such a choice of r is possible by the continuity of the g_{n-k+1}'s and by the fact that $\lim a_k = 0$. Thus by (2)

$$g_{n-k+1}(b_k, \cdots, b_n, b_{n+1}) - g_{n-k}(b_k, \cdots, b_n) \in V_{n+1}, \tag{4}$$

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