EXACT ASYMPTOTIC ESTIMATES OF BROWNIAN PATH VARIATION

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1. Summary. A standard Brownian motion in R^d, the Euclidean space of d dimensions, will be denoted by $X(t, \omega)$ or $X(t)$ when we do not need to be explicit about the point ω in the underlying probability space. We consider sample paths, that is, properties of $X(t, \omega)$ as a function of t for fixed ω. It is well known that almost all paths $X(t)$ are everywhere continuous but not of bounded variation on finite intervals and that the square variation is "almost" finite on a fixed interval and becomes constant under suitable restrictions. To be precise, if

$$\pi_n = \{0 = t_{n,0} < y_{n,1} < \cdots < t_{n,k_n} = 1\}$$

is a fixed sequence of partitions of [0, 1] such that π_{n+1} contains all the division points of π_n and

$$\sigma(\pi_n) = \max_i (t_{n,i} - t_{n,i-1}) \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} |X(t_{n,i}) - X(t_{n,i-1})|^2 = d \tag{1.1}$$

with probability 1. This result is due to Lévy [11] but it can be most easily proved by a martingale argument (see Doob [4]).

If $Q(\delta)$ denotes the class of partitions π of (0, 1) such that $\sigma(\pi) < \delta$, we could ask about the size of $V_2(X, \pi)$ as π ranges over $Q(\delta)$, where

$$V_2(X, \pi) = \sum_{i=1}^{k} |X(t_i) - X(t_{i-1})|^2$$

when $\pi = \{0 = t_0 < t_1 < \cdots < t_k = 1\}$. Lévy showed in [12] that $V_2(X, \pi)$ was unbounded for $\pi \in Q(\delta)$ while the corresponding $V_\alpha(X, \pi)$ for $\alpha > 2$ remains bounded with probability 1. The modulus of continuity for Brownian motion (see Lemma 2.1) leads easily to a proof that

$$V_\psi(X, \pi) = \sum_{i=1}^{k} \psi(|X(t_i) - X(t_{i-1})|)$$

is bounded for all π whenever $\psi(s) = s^2/\log^* s$, where $\log^* s = \max\{1, |\log s|\}$. Although this function $\psi(s)$ is bigger as $s \to 0$ than s^α for any $\alpha > 2$, it is not

Received December 20, 1971.