ON A COMBINATORIAL IDENTITY OF WINQUIST AND ITS GENERALIZATION

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1. Introduction and notation. L. Winquist [7] has recently established the following identity.

$$\begin{aligned}(1.1)\qquad &\prod_{n=1}^{\infty} (1 - x^n)^2(1 - ax^{n-1})(1 - bx^{n-1})(1 - ab^{-1}x^{n-1})(1 - abx^{n-1})\\ &\qquad\cdot(1 - a^{-1}x^n)(1 - b^{-1}x^n)(1 - a^{-1}bx^n)(1 - a^{-1}b^{-1}x^n)\\ &= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j}\{(a^{-3i} - a^{3i+3})(b^{-3j} - b^{3j+1})\\ &\qquad + (a^{-3j+1} - a^{3j+2})(b^{3i+2} - b^{-3i-1})\}x^{\frac{3}{2}i(i+1)+\frac{1}{2}j(3j+1)}.\end{aligned}$$

He used this identity to derive some congruences and other properties associated with the partition function $p(n)$.

In this paper we give a very simple proof of this identity using only the triple product identity of Jacobi

$$(1.2)\qquad \prod_{n=1}^{\infty} (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1}) = \sum_{n=-\infty}^{\infty} (-1)^n a^n x^{n^2}.$$

(For a simple proof of this identity refer to G. E. Andrews [1].)

Later in the paper we also indicate a combinatorial property implied by this identity.

Our proof of (1.1), which differs from that of Winquist, is perhaps the natural one. Further, it suggests a generalization of Winquist's identity, and this is given in §3. Our method of proof is capable of wider applications. For instance, we used it elsewhere to obtain [3] a very simple proof of the quintuple product identity [4], [5] and [6]

$$\begin{aligned}&\prod_{n=1}^{\infty} (1 - x^n)(1 - ax^n)(1 - a^{-1}x^{n-1})(1 - a^2x^{2n-1})(1 - a^{-2}x^{2n-1})\\ &\qquad\qquad = \sum_{n=-\infty}^{\infty} (a^{3n} - a^{-3n-1})x^{\frac{1}{2}(3n^2+n)}.\end{aligned}$$

It is necessary to introduce some notation to minimize the need for writing long expressions. Throughout the paper we adopt the following notation.

Unless otherwise stated, all summations are from $-\infty$ to ∞ and all products are from 1 to ∞.

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