IMBEDDING OF COMPACT METRIC SPACES INTO CUBES

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1. Introduction. The present paper grew from an attempt to settle the following unsolved problem for the case where the k-dimensional set X is compact. "It is not known $\cdots$ whether there exists any k-dimensional subset of E_n containing a topological image of every k-dimensional subset of E_n when $n < 2k + 1$" [4; 65].

S. W. Hahn [3] has shown that when $k = n$ and $k = n - 1$, M_n^k contains a strong topological image (image under a homeomorphism of the space onto itself) of every k-dimensional subset of E_n. He also showed, by counterexample, that the result is false for $0 \leq k \leq n - 2$ for general X. Sharp [6] added the hypothesis that X is an F_σ-set in E_n and proved that X can be imbedded in M_n^k under a strong homeomorphism if $k = n - 2$ and $n \geq 4$. He also showed that this result fails for $0 \leq k \leq n - 3$ and for $k = 1$ if $n = 3$.

The problem which motivated the present study remains unsolved, but the attempts to solve it lead to interesting related problems. Now if $n \geq 2k + 1$, it is known that the set of homeomorphisms of X into M_n^k is a dense G_σ-set in the set $H(X, I_n)$ of all homeomorphisms of X into I_n. Using this as a model we formulate the following proposition, which if true, would give an affirmative answer to the original problem.

(P): Suppose X is compact and $\dim X = k$. Then the set of homeomorphisms $\{f\}$ such that $f(X) \subset M_n^k$ is a dense G_δ-set in the function space $H(X, I_n)$.

The present paper formulates three new propositions and shows that each is equivalent to proposition (P), that is, if any one of these is true, then all four are true. The precise statement is given as Theorem 4.1. Even if proposition (P) is false in general, there certainly exist compact spaces X for which it is true and for such an X the four propositions of Theorem 4.1 are all true.

We remark that proposition (P) is true for cases $k = n - 1$, $k = n - 2$, and $n \geq 2k + 1$ by Lemma 4.2, a theorem of Sharp [6; 561], and a theorem of Hurewicz and Wallman [4; 64] respectively. The first unsolved case is when $k = 3$ and $n = 6$.

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2. Preliminary lemmas. Throughout the paper Euclidean n-space E_n and an n-dimensional cube I_n in E_n are metrized by the Euclidean metric denoted by d. The sets M_n^j and L_n^j consist of the totality of points in I_n with at most j and with at least j rational coordinates respectively. Note that $\dim M_n^j = j$ and $\dim L_n^j = n - j$.

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