A GENERATING FUNCTION AND A BINOMIAL IDENTITY

BY J. P. SINGHAL

1. Introduction. It is well known that the classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies the generating function [6; 69]

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1 - t + \rho)^{-\alpha}(1 + t + \rho)^{-\beta}, \tag{1.1}$$

where

$$\rho = (1 - 2xt + t^2)^{\frac{1}{2}}.$$

It is not hard to see that (1.1) is equivalent to the more symmetrical form

$$\sum_{j,k=0}^{\infty} \binom{\alpha + j + k}{j}\binom{\beta + j + k}{k} u^j v^k = 2^{\alpha+\beta}\sigma^{-1}(1 - u + v + \sigma)^{-\alpha}(1 + u - v + \sigma)^{-\beta}, \tag{1.2}$$

where

$$\sigma = \{(1 - u - v)^2 - 4uv\}^{\frac{1}{2}}.$$

A simple and direct proof of (1.2) is due to Carlitz [2] (see also [5]).

Recently Carlitz [4] gave the following two extensions of (1.2).

$$\begin{aligned}\sum_{i,j,k=0}^{\infty} &\binom{\alpha + i + j}{i}\binom{\beta + j + k}{j}\binom{\gamma + k + i}{k} u^i v^j w^k \\ &= 2^{\alpha+\beta+\gamma}R^{-1}\left(\frac{1 - w}{1 - u + v - w + R}\right)^{\alpha}\left(\frac{1 - u}{1 - u - v + w + R}\right)^{\beta} \\ &\quad\cdot\left(\frac{1 - v}{1 + u - v - w + R}\right)^{\gamma},\end{aligned} \tag{1.3}$$

where

$$R = \{(1 - u - v - w)^2 - 4uvw\}^{\frac{1}{2}}.$$

$$\sum_{i,j,k,r=0}^{\infty} \binom{\alpha + i + j}{i}\binom{\beta + j + k}{j}\binom{\gamma + k + r}{k}\binom{\delta + r + i}{r} u^i v^j w^k t^r = S^{-1}(1 - x)^{\alpha}(1 - y)^{\beta}(1 - z)^{\gamma}(1 - s)^{\delta}, \tag{1.4}$$

where

$$S = \{(1 - u - v - w - t + uw + vt)^2 - 4uvwt\}^{\frac{1}{2}}$$

Received March 13, 1970. Revision received May 3, 1971.