THE GROWTH OF THE MAXIMUM MODULUS OF UNIVALENT FUNCTIONS

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1. Introduction. We denote by $\mathcal{S}$ the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in $U = \{z \mid |z| < 1\}$; $\mathcal{S}^*$ and K will denote the subclass of $\mathcal{S}$ for which the image of U is, respectively, starlike with respect to the origin and close-to-convex.

In this paper we discuss the rate of growth of

$$M(r, f) = \max_{|z|=r} |f(z)|$$

for functions in $\mathcal{S}$, $\mathcal{S}^*$ and K. There are two well known theorems which illustrate that under certain hypotheses, $M(r, f)$ has a very regular growth rate.

Theorem A (Hayman [1; 9]). *If $f \in \mathcal{S}$, then $r^{-1}(1 - r)^2 M(r, f)$ is a decreasing function of r on $(0, 1)$ and hence tends to a limit as r increases to 1.*

In particular Theorem A implies that if

$$\limsup_{r \to 1} (1 - r)^2 M(r, f) = \gamma > 0, \tag{1.1}$$

then

$$M(r, f) > \frac{r\gamma}{(1 - r)^2} \qquad (0 < r < 1). \tag{1.2}$$

Theorem B (Pommerenke [6]). *If $f \in \mathcal{S}^*$, then*

$$\lim_{r \to 1} \frac{\log M(r, f)}{\log \dfrac{1}{1 - r}} = \lambda$$

exists.

It follows from Theorem B that if $0 \leq \lambda \leq 2$ and

$$\limsup_{r \to 1} (1 - r)^{\lambda} M(r, f) = \gamma > 0,$$

then there exists a constant $A > 0$ such that

$$M(r, f) > \frac{A}{(1 - r)^{\lambda}} \qquad (0 < r < 1).$$

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