POSITIVE-DEFINITE FUNCTIONS ON INVOLUTION SEMIGROUPS

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Introduction. Considerable attention has been devoted to the study of semigroups of operators on a Hilbert space (cf. [9; Chapter XXII]). In the present paper we develop a theory which demonstrates that much of the discussion of "involution semigroups" can be reduced to a study of semigroups of the above type. B. Sz-Nagy [18; 120] seems to have been the first to have considered involution semigroups (*-semigroups); his investigations centered around operator-valued functions of positive type. We generalize the concept of (complex-valued) positive-definite functions on a group to what we call *-definite functions on a *-semigroup. In the case where the semigroup has no identity we consider only those *-definite functions which are "extendible". Much of the classical theory of positive-definite functions and group representations carries over in our setting. If S is a semitopological *-semigroup, we find a *-representation of S into the bounded operators on a Hilbert space such that the continuous extendible *-definite functions coincide with certain representation functions (Theorem 3.2).

In §4 we show how positive-definite representations of a commutative semitopological semigroup with identity are related to the completely monotonic functions (cf. [1]). New characterizations of continuous completely monotonic functions are given in Theorem 4.3 and Corollary 4.4. Moreover, it is seen that these functions form a subclass of the *-definite functions on S when the identity involution is imposed.

§5 is devoted primarily to an investigation of the *-semigroup compactifications induced by extendible *-definite functions. It is shown that a unique invariant mean exists on the algebra of functions generated by the extendible *-definite functions (Theorem 5.7).

Throughout this paper we use certain basic results and definitions from Banach algebra theory which can be found, for example, in [14].

1. Definitions and notations. Let S be a semigroup which admits a map * from S into S satisfying

(i) $(x^*)^* = x \qquad x \in S$

(ii) $(xy)^* = y^*x^* \qquad x, y \in S$.

Such an S is called an *involution* (or *-) *semigroup* with involution *. An involution can be introduced on an arbitrary commutative semigroup S by defining $x^* = x$ for all $x \in S$. Not all semigroups admit an involution as is seen by considering any set S with the right-trivial multiplication (i.e., $xy = x$ for $x, y \in S$).

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