# A GENERALIZED INFINITE PRODUCT MEASURE

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**1. Introduction.** The standard definition of the product measure in a product of infinitely many measure spaces suffers from the defect that a product of measurable sets need not be measurable (see reference [2]). In this paper we define an extension of the standard product measure which is free of this defect. However, our extended product measure is not always a regular measure when the given measures are regular. Instead it is semi-regular; this means that measurable sets differ (in the sense of symmetric difference) from open sets by sets of arbitrarily small measure. In particular our theory applies to subsets of a Tychonoff cube of arbitrary cardinality.

**2. Definitions and notation.** By a *measure space* we mean a triple $(S, \mathfrak{M}, m)$ consisting of a non-empty set $S$, a $\sigma$-field $\mathfrak{M}$ of subsets of $S$ and a countably additive measure $m$ defined on $\mathfrak{M}$. Throughout this paper $(X_\alpha, \mathfrak{M}_\alpha, \mu_\alpha)$ will denote a measure space for each $\alpha$ contained in an infinite index set $\Gamma$ and $\mu_\alpha(X_\alpha) = 1$ for every $\alpha \varepsilon \Gamma$.

Let $X = \prod_{\alpha \varepsilon \Gamma} X_\alpha$ denote the cartesian product of the collection $\{X_\alpha : \alpha \varepsilon \Gamma\}$. A subset $A$ of $X$ is called a measurable rectangle if it is of the form

$$A = \prod_{\alpha \varepsilon \Gamma} A_\alpha , \tag{1}$$

where $A_\alpha \varepsilon \mathfrak{M}_\alpha$ and, with a finite number of exceptions, $A_\alpha = X_\alpha$. If $\mu$ is a countably additive measure defined on a $\sigma$-field of subsets of $X$ containing all measurable rectangles such that $\mu(A) = \prod_{i=1}^{n} \mu_{\alpha_i}(A_{\alpha_i})$ for every measurable rectangle $A = \prod_{\alpha \varepsilon \Gamma} A_\alpha$, where $\alpha_i$ are those indices for which $A_{\alpha_i} \neq X_{\alpha_i}$, then $\mu$ is called an *infinite product measure.* Let $\mathfrak{B}$ be the $\sigma$-field of subsets of $X$ generated by the collection of all measurable rectangles. We shall let $\mu$ denote the unique infinite product measure defined on $\mathfrak{B}$ whose existence is established in [2]. $(X, \bar{\mathfrak{B}}, \bar{\mu})$ will denote the completion of $(X, \mathfrak{B}, \mu)$.

Now let $S$ be an arbitrary set. If $A$ and $B$ are subsets of $S$, the symmetric difference of $A$ and $B$ is denoted by $A + B$, i.e., $A + B = (A - B) \cup (B - A) = (A \cap B') \cup (A' \cap B)$, where $A'$ denotes the complement of $A$ relative to $S$. A *$\sigma$-ideal* of subsets of $S$ is a collection of subsets $\mathfrak{D}$ of $S$ such that $A \varepsilon \mathfrak{D}$ and $B \subseteq A$ imply $B \varepsilon \mathfrak{D}$, and $A_i \varepsilon \mathfrak{D}$, $i = 1, 2, \cdots$, imply $\bigcup_{i=1}^{\infty} A_i \varepsilon \mathfrak{D}$.

**3. An extension of $\bar{\mu}$.** In the sequel we shall assume that $\Gamma$ is uncountable. A subset $A$ of $X = \prod_{\alpha \varepsilon \Gamma} X_\alpha$ is said to *depend on a countable number of coordinates* if there exists a countable subset $J$ of $\Gamma$ such that a point $(x_\alpha)_{\alpha \varepsilon \Gamma}$ of $X$ belongs

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