ON THE DISTRIBUTION OF GENERALIZED K-FREE INTEGERS IN RESIDUE CLASSES

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1. Introduction and notation. Throughout this paper, the letters a, b, d, e, h, K, α, γ, q, r, s, m, n represent natural numbers and p is reserved for primes. Let $\zeta(s)$ denote the Riemann zeta function. We recall that an integer n is said to be K-free whenever it is not divisible by the K-th power of any prime. In the extreme cases, unity is the only 1-free integer, and every integer is ∞-free.

There is a vast literature concerning K-free integers for general and special values of K. We might refer to the bibliography given in Eckford Cohen and Richard L. Robinson [1] in this connection.

In this paper we wish to generalize the notion of K-free integers as follows. Let q and r be fixed integers such that $0 < q < r$. Any integer $n > 1$ has the unique representation

$$n = a^r b, \tag{1.1}$$

where b is r-free. We shall call a^r the r-th power part of n and b the r-free part. If, in addition, b is q-free, we shall call n a generalized q-free integer or an (r, q)-integer. Notice that in the limiting case when $r \to \infty$, an (r, q)-integer becomes a q-free integer. We may also observe that an $(r, 1)$-integer is the same as an r-th power integer.

The (r, q)-integers were first introduced in 1966 in a paper by M. V. Subbarao and V. C. Harris [5] in connection with a generalization of the well known Ramanujan trigonometric sum $C(n, r)$. In that paper the following result was established. Let S be the set of all (r, q)-integers. Note that S contains unity. Let $\lambda(n) = \lambda_{r,q}(n)$ be the multiplicative arithmetic function defined by

$$\lambda(p^a) = \begin{cases} 1 & a \equiv 0 \pmod r \\ -1 & a \equiv q \pmod r \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(rs)}{\zeta(qs)} \qquad \left(s > \frac{1}{q}\right). \tag{1.3}$$

Also

$$\psi(n) = \psi_{r,q}(n) = \sum_{d \mid n} \lambda(d) = \begin{cases} 1 & n \in S \\ 0 & n \notin S. \end{cases} \tag{1.4}$$

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