# ENUMERATION OF SYMMETRIC ARRAYS, II

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**1. Introduction and summary.** $S_n(r)$ denotes the number of $n \times n$ arrays $(a_{ij})$, where the $a_{ij}$ are nonnegative integers such that

$$a_{ij} = a_{ji} \qquad (i, j = 1, 2, \cdots, n) \tag{1.1}$$

and

$$\sum_{i=1}^{n} a_{ij} = r \qquad (j = 1, 2, \cdots, n). \tag{1.2}$$

In an earlier paper [1], the writer showed that

$$\sum_{r=0}^{\infty} S_3(r)x^r = \frac{1 + x + x^2}{(1 - x)^4(1 + x)} \tag{1.3}$$

and

$$\sum_{r=0}^{\infty} S_4(r)x^r = \frac{1 + 4x + 10x^2 + 4x^3 + x^4}{(1 - x)^7(1 + x)}. \tag{1.4}$$

In proving (1.4) it was found convenient to consider the following related problems. Let $T(r, s)$ denote the number of quadruplets $(b, c, d, e)$, where $b, c, d, e$ are nonnegative integers such that

$$b + c \le r, d + e \le r, b + d \le s, c + e \le s. \tag{1.5}$$

Then

$$\sum_{r,s=0}^{\infty} T(r, s)x^r y^s = \frac{1 + 4xy + 4x^2y^2 + x^2y + xy^2 + x^3y^3}{(1 - x)(1 - y)(1 - xy)^2(1 - x^2y)(1 - xy^2)} \tag{1.6}$$

$$\sum_{r=0}^{\infty} T(r, r)x^r = \frac{(1 + x)^2}{(1 - x)^5}. \tag{1.7}$$

In the present paper we consider first the weighted sum

$$S_3(r; \lambda_1, \lambda_2, \lambda_3) = \sum_{\substack{a+b \le r \\ b+c \le r \\ a+2b+c \ge r}} \lambda_1^a \lambda_2^b \lambda_3^c, \tag{1.8}$$

which reduces to $S_3(r)$ when $\lambda_1 = \lambda_2 = \lambda_3 = 1$. We show that

$$\sum_{r=0}^{\infty} S_3(r; \lambda_1, \lambda_2, \lambda_3)x^r = \frac{1 - \lambda_1\lambda_2\lambda_3 x^3}{(1 - \lambda_1 x)(1 - \lambda_2 x)(1 - \lambda_3 x)(1 - \lambda_1\lambda_3 x)(1 - \lambda_2 x^2)}. \tag{1.9}$$

Received January 15, 1970. Supported in part by NSF grant GP-7855.