SOME RESULTS ON THE ENUMERATION OF SYMMETRIC ARRAYS

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1. Let $S(r_1, \cdots, r_n)$ be the number of $n \times n$ symmetric arrays $[a_{ij}]$, where the a_{ij} are nonnegative integers satisfying

$$\sum_{j=1}^{n} a_{ij} = r_i, \qquad i = 1(1)n.$$

It is convenient to let $S_n(r) = S(r, r, \cdots, r)$ where there are n r's.

L. Carlitz [1] found formulas and generating functions for $S_n(1)$ and $S_i(r)$, $i \leq 4$. H. Gupta [2] gave a recurrence for $S_n(2)$ and determined the generating function. D. P. Roselle [4] obtained a recurrence for $S(r_1, \cdots, r_n)$.

The object of this paper is to exhibit results for other cases of this enumeration problem.

2. Since $S(r_1, \cdots, r_n) = S(r_{i_1}, \cdots, r_{i_n})$, where $(i_1, \cdots, i_n)$ is any permutation of $(1, \cdots, n)$, we may assume $r_n \leq \cdots \leq r_1$. Then

$$S(r_1, \cdots, r_n) = \sum_A 1 = \sum_B \sum_C 1,$$

where A is

$$\begin{aligned} a_{11} + a_{12} + \cdots + a_{1n} &= r_1 \\ a_{12} + a_{22} + \cdots + a_{2n} &= r_2 \\ &\cdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} &= r_n, \end{aligned}$$

where B is

$$a_{1n} + a_{2n} + \cdots + a_{n-1\,n} \leq r_n,$$

and where C is

$$\begin{aligned} a_{11} + a_{12} + \cdots + a_{1\,n-1} &= r_1 - a_{1n} \\ a_{12} + a_{22} + \cdots + a_{2\,n-1} &= r_2 - a_{2n} \\ &\cdots \\ a_{1\,n-1} + a_{2\,n-1} + \cdots + a_{n-1\,n-1} &= r_{n-1} - a_{n-1\,n}. \end{aligned}$$

This gives the recurrence

$$S(r_1, \cdots, r_n) = \sum_{a_1 + \cdots + a_{n-1} \leq r_n} S(r_1 - a_1, \cdots, r_{n-1} - a_{n-1}). \tag{2.1}$$

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