ON THE ENUMERATION OF SYMMETRIC MATRICES

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1. Roselle [2] has found that $S(r_1, \cdots, r_n)$ the number of $n \times n$ symmetric matrices with non-negative integral elements such that the sum of the elements in the j-th row is r_j (≥ 0) is the coefficient of $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ in the product

$$G(n) = \prod_{0 \leq j < k \leq n} (1 - x_j x_k)^{-1}, \qquad x_0 = 1.$$

He has also obtained a recurrence for $S(r_1, \cdots, r_n)$.

Here we give another recurrence which appears to be simpler than his and obtain an explicit expression for $S(r_1, r_2, r_3)$.

Since the particular order in which $r_1, r_2, \cdots, r_n$ appear in $S(r_1, r_2, \cdots, r_n)$ does not affect its value, we can assume that

$$r_1 \geq r_2 \geq \cdots \geq r_n \geq 0.$$

2. Evidently

$$G(n+1) \prod_{j=0}^{n} (1 - x_j x_{n+1}) = G(n).$$

Define the operator z_j by the relation

$$z_j S(r_1, \cdots, r_j, \cdots, r_{n+1}) = S(r_1, \cdots, r_j - 1, \cdots, r_{n+1})$$

with z_0 as the identity operator. Then our recurrence is

$$\left\{ \prod_{j=0}^{n} (z_0 - z_j z_{n+1}) \right\} S(r_1, \cdots, r_n, r_{n+1}) = 0.$$

For $n = 2$, we have

$$(z_0 - z_3)(z_0 - z_1 z_3)(z_0 - z_2 z_3) S(r_1, r_2, r_3) = 0.$$

This gives

$$\begin{aligned} S(r_1, r_2, r_3) = {} & S(r_1, r_2, r_3 - 1) + S(r_1, r_2 - 1, r_3 - 1) \\ & + S(r_1 - 1, r_2, r_3 - 1) - S(r_1, r_2 - 1, r_3 - 2) \\ & - S(r_1 - 1, r_2, r_3 - 2) - S(r_1 - 1, r_2 - 1, r_3 - 2) \\ & + S(r_1 - 1, r_2 - 1, r_3 - 3). \end{aligned}$$

Since $S(r_1, 0, 0) = S(r_1) = 1$ for all $r_1 \geq 0$ and $S(r_1, r_2, 0) = S(r_1, r_2) = r_2 + 1$ for $r_1 \geq r_2 \geq 0$, giving to r_3 the values $1, 2, 3, \cdots$ in succession

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