GENERIC COVERS AND DIMENSION

By David Booth

1. Introduction. In his paper [2], K. Gödel has listed several suspicious propositions of point set theory which are implied by the continuum hypothesis. This paper concerns one of these propositions: there is an infinite dimensional subset of Hilbert space each of whose subsets is either countable or infinite dimensional. One may as well suppose that such a set H is embedded in the Hilbert cube, $I^{\aleph_0}$, where I is $[0, 1]$ taken as a topological space.

There are several popular definitions of dimension which all prove to be equivalent for separable spaces. Here we shall use the 'big inductive' dimension which gives $\dim X = -1$ if $X = 0$ and $\dim X \leq n$ when for any two disjoint closed sets A and B in X there is an open set U for which $U \supseteq A$, $U \cap B = 0$, and $\dim (\delta U) \leq n - 1$. The symbol δU denotes the topological boundary of U. Obviously I is the union of $2^{\aleph_0}$ closed sets (points) each having dimension zero. An infinite dimensional space is one which does not have finite dimension.

2. Generic covers. If $\mathcal{P} = \langle P, \leq \rangle$ is a partial order, then $D \subseteq P$ is *dense* in $\mathcal{P}$ if for each $x \in P$ there is a $y \in D$ with $y \leq x$. A subset G of P is *generic* for $\mathfrak{D}$, where $\mathfrak{D}$ is a collection of dense subsets of P, if

(i) for each $x, y \in G$ there is a $z \in G$ with $z \leq x, y$
(ii) $y \in G$ whenever $y \geq x \in G$
(iii) if $D \in \mathfrak{D}$ then $G \cap D \neq 0$.

In the theorem below X is a separable metric space, A and B are disjoint closed sets in X, and $\{F_\alpha : \alpha \in \lambda\}$ is a closed covering of X having power λ and such that $\dim F_\alpha \leq 0$. We shall strive to separate A and B by an open set with empty boundary.

Theorem 1. *When X, A, B and $\{F_\alpha : \alpha \in \lambda\}$ are as above, there is a countable partial order $\mathcal{P}(X, A, B)$ and a collection, $\mathfrak{D}$, of λ dense subsets of $\mathcal{P}(X, A, B)$ such that given any $\mathfrak{D}$ generic set, G, there is an open and closed set U, constructible from G, with $A \subseteq U$ and $B \cap U = 0$.*

Proof. One may treat such an X as if it were a subspace of $I^{\aleph_0}$ and take basic neighborhoods in $I^{\aleph_0}$ to be finite sequences of open intervals with rational endpoints. Let $P(X, A, B)$ consist of pairs of sequences of the form

$$\langle \langle N_0, \cdots, N_{n-1} \rangle, \quad \langle M_0, \cdots, M_{n-1} \rangle \rangle$$

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