A FORMAL POWER SERIES OPERATIONAL CALCULUS FOR QUASI-NILPOTENT OPERATORS

By Sandy Grabiner

1. Introduction. Suppose that T is a quasi-nilpotent, but not nilpotent, operator on a Banach space E (i.e., $\lim ||T^n||^{1/n} = 0$, but no $T^n = 0$). Since T has $\{0\}$ spectrum, one can use the Reisz operational calculus [8; 431–433] to define the operator $f(T)$ whenever f is a complex function analytic at the origin. This constructs an isomorphism between the algebra of complex formal power series with positive radius of convergence and an algebra of operators on E.

In this paper we examine the consequences of no longer insisting that the power series have positive radius of convergence. That is, we consider the set B of all complex formal power series $f = \sum_{n=1}^{\infty} \lambda_n z^n$ for which the series $\sum_{n=1}^{\infty} \lambda_n T^n$, which we denote by $\bar{f}(T)$, converges in some specified operator topology. If $\bar{f}(T)$ converges, we will let $f(T)$ be its sum. For technical reasons we have included in B only power series with zero constant terms; hence the following definition will be convenient.

Definition (1.1). A *proper formal power series* is a complex formal power series with zero constant term. A *proper algebra* is an algebra of proper formal power series which contains z, and hence all polynomials.

If A is an algebra without identity, we will denote by $A^{\#}$ the algebra formed by formally adjoining an identity to A [7; 2–3]. Similarly, if B is a vector space of proper power series, $B^{\#}$ is the sum of the vector space B and the vector space of constant formal power series.

The following definition lists the most important properties that we would like to prove for sets of converging power series and for the map $f \to f(T)$.

Definition (1.2). Suppose that T is a quasi-nilpotent, but not nilpotent, operator and that B is the set of all proper power series, f, for which $\bar{f}(T)$ converges in the strong operator topology; then T is *formally representable* in the strong operator topology if

(A1) B is a proper algebra

(A2) B is *analytically closed*, in the sense that $g \circ f \in B^{\#}$ for all f in B and all g with positive radius of convergence

(B1) the map $f \to f(T)$ is an algebra homomorphism

Received November 7, 1969. Revision received July 2, 1970. Many of the results in this paper are contained in the author's thesis completed at Harvard University under the direction of Professors L. H. Loomis and A. Gleason. The author wishes to thank Professors Loomis and Gleason for their aid and encouragement. A preliminary form of this paper was delivered to an analysis seminar at M.I.T.