ON BASES FOR THE SET OF INTEGERS

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1. Introduction. Let A be a set of m distinct integers with $0 \in A$ and $m \geq 2$. The notions of simple and non-simple A-bases were introduced by de Bruijn in a paper [2] in which he discusses the case $A = \{0, 1\}$ in some detail. For this special case, he also introduces the notion of a basic sequence giving some necessary and some sufficient conditions that a sequence be basic and giving special attention to periodic basic sequences of period 2. This latter discussion is continued in [3]. In the present paper, we generalize the notion of an A-base to that of an $\mathcal{A}$-base where $\mathcal{A} = \{A_i\}_{i \geq 1}$ and each A_i is a set of m_i distinct integers with $0 \in A_i$ and $m_i \geq 2$ for all i. The structure of $\mathcal{A}$-bases is studied and rather general methods of constructing simple and non-simple $\mathcal{A}$-bases are given.

We begin by introducing the necessary definitions and notation.

Definition 1. Let $\mathcal{A} = \{A_i\}_{i \geq 1}$ where the A_i are as above. The integral sequence $B = \{b_i\}_{i \geq 1}$ is called an $\mathcal{A}$-base provided that every integer n can be represented uniquely in the form

$$n = \sum_{i=1}^{r(n)} a_i b_i \ , \ a_i \in A_i \quad \forall i.$$

If B can be written (with possible rearrangement) in the form $B = \{d_i M_{i-1}\}_{i \geq 1}$ where the d_i are integers and where $M_0 = 1$ and $M_i = \prod_{j=1}^{i} m_j$ for $i \geq 1$, then it is called a simple $\mathcal{A}$-base.

Definition 2. If the sequence $B = \{b_i\}_{i \geq 1}$ is an $\mathcal{A}$-base and $A_i = A$ for all i, then B is called an A-base.

Finally, if m is an integer and A is a set of integers, by mA we mean the set $S = \{s \mid s = ma, a \in A\}$.

2. Simple $\mathcal{A}$-bases. The fact that simple $\mathcal{A}$-bases exist is an immediate consequence of the fact that every integer n can be represented uniquely in the form

$$n = \sum_{i=1}^{r(n)} (-1)^{s_i} a_i M_{i-1} \ , \ 0 \leq a_i < m_i \quad \text{for all} \quad i \tag{1}$$

where m_i and M_i are as above for all i and $\{s_i\}_{i \geq 1}$ is a sequence of zeros and ones containing infinitely many of each. That such representations exist seems first to have been proved by J. L. Brown [1].

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