ON FINITE EQUIVALENT INVARIANT MEASURES FOR SEMIGROUPS OF TRANSFORMATIONS

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Introduction. It has been shown by L. Sucheston in an interesting paper [18] among others, that if $(X, \mathcal{S}, p)$ is a finite measure space, $s: X \to X$ a measurable transformation and M the support functional of the set of Banach limits on the space of bounded real sequences on the positive integers, then the following is a necessary and sufficient condition for the existence of a finite invariant measure on $\mathcal{S}$, equivalent to p: $p(A) = 0$ implies $p(s^{-1}A) = 0$, and $p(A) > 0$ implies $M(p(s^{-n}A)) > 0$. Furthermore, Sucheston identifies the functional M by

$$M(\{x_n\}) = \lim_{n\to\infty} \sup_j \sum_{i=j}^{n+j-1} x_i \ .$$

It has been shown by Blum and Friedman [2] among other results, that if $(X, \mathcal{S}, p)$ is a finite measure space and if S is a *countable* class of *commuting* measurable transformations $s: X \to X$ such that each $s \in S$ admits an s invariant finite measure μ_s on $\mathcal{S}$, equivalent to p, and $p(A) > 0$ implies $\inf\{p(s^{-1}A);\ s \in S^*\} > 0$ where S^* is the semigroup generated by S, then there exists a finite S-invariant measure μ equivalent to p.

We prove in this paper some theorems which generalize the main results of Sucheston in [14] to all amenable semigroups (i.e. which admit a translation invariant mean on the space of bounded real functions on S) including the identification of M given in [14]. This result when applied to the case when S is a commutative semigroup (not necessarily countable) yields a result which is much stronger than that of Blum and Friedman. (It reduces to the interesting result of Sucheston in the case of a one transformation semigroup, while the blanket assumption made in [2] is that each transformation s admits a finite s-invariant measure μ_s equivalent to the given one.)

In the second part of the paper we identify the support functional of the set of Banach limits on any semigroup in terms of nets of finite means converging in norm to left invariance (Theorem 6). We strengthen in the end the main results for the case that S is an extremely amenable semigroup.

The reader familiar with Sucheston's paper [14] will recognize our indebtedness in §1 to the ideas and proofs included in that paper. Sucheston's results in [14] constitute significant improvements on results of Y. N. Dowker [18], [19], Calderón [17] and Hajian–Kakutani [11].

Notations. In all that follows we assume that $(X, \mathcal{S}, p, S)$ is given, where $\mathcal{S}$ is a σ-algebra of subsets of the nonvoid set X, p is a finite nontrivial measure

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