ON TRANSITIVE REPRESENTATIONS OF SEMIGROUPS BY 1-1 PARTIAL TRANSFORMATIONS

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1. Introduction and summary. The most natural representations of semigroups are those by partial or full transformations on a set. Among these the most fruitful have been representations by 1-1 partial transformations. They are particularly suitable for representing inverse semigroups and those which can be embedded in inverse semigroups.

Every symmetric representation by 1-1 partial transformations is, up to equivalence, uniquely a sum of transitive and null representations, so the problem of finding all such representations can be restricted to finding the transitive ones. The problem is further reduced, for an arbitrary semigroup S, to the determination of all proper unitary strong subsemigroups of S. For an inverse semigroup S, every representation by 1-1 partial transformations is symmetric, every unitary subsemigroup is strong, and the theory of transitive representations of S is considerably ramified. These results are due to Shain [6], [7], [8]; an account of it is given in [1, §§11.4, 7.3]. On the other hand, Ponizovski has studied transitive representations by partial transformations of semigroups satisfying certain minimality conditions, and in [4] constructed directly all representations by 1-1 partial transformations of inverse semigroups with the minimal condition on principal right ideals. Another approach for inverse semigroups has been devised by Reilly [5] (see [1], §7.6).

The purpose of this paper is the construction of a class of proper unitary strong subsemigroups of a semigroup S, which can be obtained from the same kind of subsemigroup of some principal factor of S. We then consider the associated representation and apply this to inverse semigroups and to various semigroups of transformations.

In §2 we introduce a minimum of necessary notation and background. §3 is devoted to a description of proper unitary strong subsemigroups H of a semigroup S which, intuitively speaking, have a "nadir" K in some $\mathcal{J}$-class; H is then the idealizer of K in S. With a certain restriction on the $\mathcal{J}$-class, we reconstruct H when K is given. The remainder of §3 deals with conjugacy of these subsemigroups and certain properties of semigroups under consideration. In §4 we apply these results to inverse semigroups; we generalize and sharpen some of Ponizovski's results [4], and find the number of inequivalent representations in question for several classes of semigroups. In §5 we apply the results of §3 to the semigroups of all (partial) transformations on a finite set and to the semigroup of $n \times n$ matrices over a division ring, and find that all subsemigroups in question are subgroups of the group of units.

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