THE FILTER CONDITION, THE CLOSED NEIGHBOURHOOD CONDITION, AND CONSISTENT SEMINORMS

By D. J. H. Garling

1. Introduction. Suppose that E is a Hausdorff topological vector space. Following Ruckle [9], a subspace X of E with a locally convex topology for which the inclusion mapping of X into E is continuous will be called an *E-space*. One situation which is of particular interest is the case where E is equal to ω, the space of all scalar (real or complex) sequences, with the usual topology of coordinatewise convergence, and where X is a locally convex ω-space containing ϕ, the space of all sequences with only finitely many non-zero terms. In this case X is called a K-space [3]. A Banach K-space is called a BK-space, a Fréchet K-space an FK-space.

The purpose of this paper is to consider the relationship between three topological conditions on an E-space X.

X is said to satisfy the *filter condition* if the following condition is satisfied:

(*) if θ is a Cauchy filter base on X, and if θ converges in E to a point x of X, then θ converges in X to x.

This condition was introduced in a rather more general setting by Wendy Robertson [8].

X is said to satisfy the *closed neighbourhood condition* if there is a base of neighbourhoods of o in X which are closed in X in the topology induced by E. This is slightly different from the definition given in [8]. The closed neighbourhood condition is considered for K-spaces in [9, §4].

The third condition involves semi-norms. An extended semi-norm p on E is said to be *consistent* if whenever $\{x^n : n = 1, 2, \cdots\}$ is a Cauchy sequence with respect to p for which $\lim_n x^n = 0$ in E, then $\lim_n p(x^n) = 0$. An E-space X is said to be *consistent* if its topology can be determined by means of a system P of consistent semi-norms. This condition was introduced by Ruckle [9].

In the next section we shall prove the following

THEOREM 1. *Suppose that X is an E-space. If X satisfies the closed neighbourhood condition, X is consistent, and X satisfies the filter condition. If E is metrizable, and if X is consistent, then X satisfies the filter condition.*

The rest of the paper is devoted to giving counterexamples to show that other implications do not hold.

2. Proof of Theorem 1. Suppose first that X satisfies the closed neighbourhood condition. The proof that X satisfies the filter condition follows [8]

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