SUPPORT POINTS IN LOCALLY CONVEX SPACES

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The main purpose of this paper is to show that support point theorems can be obtained for closed bounded convex sets in complete locally convex spaces which parallel some of the theorems of Bishop and Phelps for Banach spaces in [2]. The basic technique is to embed the convex set in question as a closed convex set in a suitable Banach space and then apply the Bishop–Phelps results. In the first section we study the situation when the containing space is a Fréchet space. At the end of the section we give an extension of a theorem of Krein–Milman type (Lindenstrauss [8], Bessaga and Pełczyński [1]) to countable products of separable dual Banach spaces.

In the second section we give some analogues of the results of the first section when the containing space is not necessarily Fréchet, but only complete and locally convex. For unbounded closed convex sets in Fréchet spaces, the situation is quite different from that in Banach spaces: in [5], Klee gave an example of an unbounded proper closed convex set in $R^{\aleph_0}$ which is not supported at any of its points by a non-trivial linear functional, continuous or not. Both Klee and Phelps [5], [10] asked whether a bounded closed convex set in a Fréchet space must be supported at one of its points by a nontrivial linear functional which is continuous on the whole space. Also in the second section, we provide an example to settle this question negatively, giving some sharpness to the results in the first section.

1. We begin with some definitions and notation. Throughout, "locally convex space" will mean "locally convex Hausdorff topological linear space"; "subspace" will mean "linear subspace". Let C be a convex subset of a real locally convex space E and let F be a subspace of E', the algebraic dual of E. An *F-support point* of C is a point x of C for which there is a non-zero f in F such that $f(x) = \sup_{c \in C} f(c)$; f is said to *support C at x*.

We shall be concerned only with three possible choices for F: E'; E_C^*, the set of all f in E' which are continuous on C; and E^*, the set of all f in E' which are continuous on E.

If $\{B_\alpha\}$ is an indexed family of Banach spaces, denote by $\mathcal{P}_{l_1} B_\alpha$ the set of elements $x = (x_\alpha)$ in $\prod_\alpha B_\alpha$ such that $\sum_\alpha \|x_\alpha\|_\alpha < \infty$, $\| \ \|_\alpha$ being the norm on B_α. Similarly, let $\mathcal{P}_m B_\alpha$ denote the set of all x in $\prod_\alpha B_\alpha$ such that $\sup_\alpha \|x_\alpha\|_\alpha < \infty$. The expressions $\sum_\alpha \|x_\alpha\|_\alpha$ and $\sup_\alpha \|x_\alpha\|_\alpha$ define norms on $\mathcal{P}_{l_1} B_\alpha$ and $\mathcal{P}_m B_\alpha$, and the following proposition about these normed spaces will be useful:

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