OPEN SIMPLICIAL MAPS OF SPHERES ON MANIFOLDS

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1. Introduction. In this paper we shall consider, up to homology, some questions raised by H. Hopf in [7, Paragraphs 2 and 3]. Let f be an open simplicial map of n-sphere on n-manifold for $n \geq 2$ and let B_f be the set of points at which f fails to be a local homeomorphism. Suppose $f \mid f^{-1}f(B_f)$ is a homeomorphism. We prove that in this case B_f is a rational $(n - r)$-homology sphere where r is even, and that each torsion coefficient is relatively prime to the degree of f. We use the fact that these maps are pseudocovering maps in the sense of Church and Hemmingsen [1, Definition 5, Theorem 2.4]. These maps may also be considered the simplest examples of Montgomery–Samelson fiberings [8]. We prove that such a map from n-manifold to n-polyhedron P can be considered simplicial provided that $[P, f(B_f)]$ is a polyhedral pair.

The idea is roughly this: If one could find a periodic homeomorphism of the n-sphere onto itself whose orbits were the sets $f^{-1}(x)$, then B_f would be its fixed-point set and could be studied by use of P. A. Smith's theory. Since it is not known that B_f is a manifold, or how it might be embedded in the n-sphere, this idea does not work. One can, however, investigate the structure of B_f by special homologies analogous to Smith's.

The author is pleased to acknowledge helpful conversations with Professor Erik Hemmingsen who made suggestions and showed the author many instructive examples. Hemmingsen's results [4], [5] on the structure of B_f are used crucially in the proof of the main theorem.

2. Special homology. In this section we define some special homology groups motivated by the work of P. A. Smith [9]. We prove a theorem about the dimension of these homology groups as vector spaces over coefficient fields motivated by the work of E. E. Floyd [3].

Definition 1. Let X be a finite simplicial complex and X_0 a subcomplex. Let $C(X)$, $C(X_0)$ and $C(X, X_0)$ be the integral chain complexes. Let ρ and τ be homomorphisms of the graded group $C(X)$ into itself. Let σ denote either ρ or τ and let $\bar{\sigma}$ denote respectively τ or ρ. Suppose

(A) $\sigma\bar{\sigma} = 0$,
(B) $\sigma[C(X_0)] = 0$,
(C) If $\sigma(c) = 0$, there is a in $C(X, X_0)$ and b in $C(X_0)$ with $c = \bar{\sigma}(a) + b$,
(D) For any c in $C(X)$, $\partial\sigma(c) - \sigma\partial(c) \in C(X_0)$,
(E) $\sigma[C(X)] \cap C(X_0) = 0$.

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