SOME REMARKS ON CONVEX MAPS OF THE UNIT DISK

By T. J. Suffridge

1. Introduction. Let K be the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are univalent in the disk $E = \{z: |z| < 1\}$ and map the disk onto convex domains. Bernardi [1] and [2] has conjectured that if $f \in K$ and $g/b_1 \in K$ where $g(z) = b_1 z + b_2 z^2 + \cdots$, $b_1 > 0$ and if g is subordinate to f(i.e. $g \prec f$), then

$$\text{(1)} \qquad \begin{aligned} &\operatorname{Re}\,[f(z)/g(z)] > 1, \\ \text{and} \quad &\operatorname{Re}\,[zf'(z)/(f(z) - g(z))] > \tfrac{1}{2}, \qquad z \in E. \end{aligned}$$

The following example shows that both parts of this conjecture are false. Choose $f(z) = z/(1 - z)$ and $g(z) = 2/\pi \tan^{-1} z$, the branch chosen so that $\tan^{-1} 0 = 0$. Then $f(E)$ is the half-plane $\operatorname{Re}(w) > -\frac{1}{2}$ and $g(E)$ is the strip $-\frac{1}{2} < \operatorname{Re} w < \frac{1}{2}$ so $g \prec f$. But $|g(z)| \to \infty$ as $z \to \pm i$ while f and f' are bounded as $z \to \pm i$. This means that if $a > 0$ and $b > 0$ then neither $\operatorname{Re}\,[f(z)/g(z)] > a$ nor $\operatorname{Re}\,[zf'(z)/(f(z) - g(z))] > b$ can hold for all $z \in E$.

In this paper we show that if $f(z) = z + a_2 z^2 + \cdots$ in E, then $f \in K$ if and only if $\operatorname{Re}\,[F(z, \zeta)] > \frac{1}{2}$ when $z, \zeta \in E$ where

$$\text{(2)} \qquad F(z, \zeta) = \begin{cases} zf'(z)/(f(z) - f(\zeta)) - \zeta/(z - \zeta) & \text{if } z \neq \zeta \\ \frac{1}{2} zf''(z)/f'(z) + 1 & \text{if } z = \zeta. \end{cases}$$

This result will imply the well-known results of Strohhäcker [4] that if $f \in K$ then $\operatorname{Re}\,[zf'(z)/f(z)] > \frac{1}{2}$ and $\operatorname{Re}\,[f(z)/z] > \frac{1}{2}$, $z \in E$. We show also that the sequence $\{1/n\}_{n=1}^{\infty}$ preserves subordination of starlike functions. More precisely, if f and g map E conformally onto convex domains with the conditions $f(0) = g(0) = 0$, $zf'(z) \prec zg'(z)$ then $f \prec g$.

2. Some properties of convex mappings.

Theorem 1. *Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in E. Then $f \in K$ if and only if*

$$\text{(3)} \qquad \operatorname{Re}\,[F(z, \zeta)] \geq \tfrac{1}{2}, \qquad z, \zeta \in E$$

where $F(z, \zeta)$ is given by (2).

Proof. We first observe that $F(z, \zeta)$ is continuous and hence analytic in both z and ζ. It is clear that (3) implies $\operatorname{Re}\,[zf''(z)/f'(z) + 1] \geq 0$, $z \in E$ and therefore (3) implies $f \in K$.

Now suppose $f \in K$. We will first show that (3) holds when $|z| = |\zeta| < 1$.

Received December 18, 1968. This research was supported by NSF Grant GP 8225.