MANIFOLDS ADMITTING A NON-HOMOTHETIC CONFORMAL TRANSFORMATION

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1. Introduction. Let M be an n-dimensional, compact, connected Riemannian manifold with (positive definite) metric tensor g. If g^* is the tensor field $\rho^2 g$ where ρ is a positive function on M, then clearly, g^* is a Riemannian metric on M which is said to be *conformally related* to g. If ρ is a constant function, the change in metric is called *homothetic.*

Denote by R the Ricci tensor of (M, g). The manifold is called an *Einstein space* if it carries an Einstein metric g, that is, if

$$R = \lambda g$$

for some scalar field λ. Let r denote the scalar curvature of (M, g), that is, let r = trace Q where Q is the Ricci operator (obtained from R and g). Then for $n > 2$, λ is a constant equal to r/n.

If M is a compact Riemannian manifold, and the scalar curvatures r and r^* of the conformally related metrics g and g^* respectively, on M are equal non-positive constants, then $g^* = g$ (see Lemma 2).

We assume throughout that M is connected and orientable. If M is not orientable, we have only to take an orientable two-fold covering space of it. We also assume that dim $M > 3$ unless otherwise specified.

Let G be the tensor measuring the deviation from an Einstein metric, that is, let

$$G = R - (r/n)g.$$

Then,

Theorem. *Let (M, g) be a compact Riemannian manifold with r = const., and let g^* be a non-homothetic conformally related metric such that $r^* = r$. Then, if*

$$(*) \qquad \int_M u^{-n+1}\langle G\, du, du\rangle\, dV \geq 0 \quad (u = 1/\rho),$$

where dV is the volume element, (M, g) is globally isometric with an Euclidean sphere, and conversely. If G (viewed as a linear transformation field) defines a positive semi-definite quadratic form on the space of exact 1-forms, the same conclusion prevails.

Observe that the condition $r^* = r$ is not necessary if the conformal change

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