EMBEDDING A TOPOLOGICAL DOMAIN IN A COUNTABLY GENERATED ALGEBRAIC RING EXTENSION

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Hinrichs has proven [3; 404, Theorem 3.4] that if $\mathfrak{I}$ is a locally bounded ring topology on a field k, and if K is an algebraic extension field of k of countable degree, i.e., K is obtained by adjoining countably many algebraic elements to k, then there is a ring topology $\mathfrak{I}_K$ on K such that $\mathfrak{I}_K \mid k = \mathfrak{I}$. Our objective here will be to generalize this result by replacing the smaller field k by an integral domain I, and replacing the larger field by a commutative ring A, which may contain zero divisors. We must add the hypothesis about $\mathfrak{I}$ on I that multiplication by nonzero elements is open, that is, for every nonzero a in I, the mapping $x \to ax$ from I into I is an open mapping for $\mathfrak{I}$. Note that a ring topology on a field necessarily has this property.

The result is obtained in two stages. In §1, we extend a topology from a field to an extension ring, modifying Hinrichs' procedure slightly. In §2 we get the result for a ring A over an integral domain I by showing that the results of §1 applied to A^* (the ring of fractions for A which contains inverses for the nonzero elements of I) will lead to the result.

Throughout, we shall use the standard characterization of a ring topology in terms of a basic system $\mathfrak{U}$ of neighborhoods of zero [1; 76]. A set B is *bounded* if for all U in $\mathfrak{U}$ there is a V in $\mathfrak{U}$ such that $BV \subseteq U$. A ring topology is *locally bounded* if there is a bounded neighborhood of zero.

1. A ring over a topologized field. Let K be a field, and let A be a ring containing K such that A can be obtained by adjoining countably many algebraic elements to K. Let $a_1, a_2, \cdots$ denote these elements. Let $A_0 = K$, and if $A_0, \cdots, A_k$ are defined, let $A_{k+1} = A_k[a_{k+1}]$. Then $A = \bigcup_{k=0}^{\infty} A_k$.

Note that A is a vector space over K. We will obtain a basis for A in the following way.

First, if a_1 is the root of a polynomial of degree r_1 in $K[x]$, then $1, a_1, a_1^2, \cdots, a_1^{r_1-1}$ form a set of vector space generators for $A_1 = K[a_1]$ over K. From this set, extract a basis for the vector space A_1, and denote this basis by $1 = b_1, b_2, \cdots, b_{s_1}$.

Assume now that we have found elements $b_1, \cdots b_{s_1}; b_{s_1+1} \cdots, b_{s_2}; \cdots; b_{s_n}$ such that for each $k \leq n$, $b_1 \cdots, b_{s_k}$ is a vector space basis for A_k over K. Now if r_{n+1} is the degree of a_{n+1} over K, then $\{1, a_{n+1}, \cdots a_{n+1}^{r_{n+1}-1}\}$ is a set of module generators for $A_{n+1} = A_n[a_{n+1}]$ over A_n. Thus, the set $\{b_i a_{n+1}^j : 1 \leq i \leq s_n,$

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