NO SEMIGROUP ON THE TWO-CELL WITH IDEMPOTENT BOUNDARY

By Edward N. Ferguson

In 1961 A. D. Wallace proposed the following question, "Does there exist a semigroup structure on the n-cell such that the set of idempotents is precisely the bounding $(n - 1)$ sphere?" We shall show such structures do not exist in the case $n = 2$; specifically:

Theorem. *There does not exist a semigroup structure on the two-cell such that the set of idempotents is precisely the bounding one sphere.*

The proof will be by contradiction. We begin with some definitions and assumed results.

A (topological) semigroup S is a Hausdorff space endowed with a continuous, associative multiplication. Standard references are [2] and [8]. In particular facts stated without attribution will be found in [8]. Particular semigroups which will be of interest are the unit interval with the usual multiplication (denoted J_1) and the interval $[\frac{1}{2}, 1]$ with the operation $x \cdot y = \max\{xy, \frac{1}{2}\}$ (denoted J_2).

Our notation is standard with the following (possible) exceptions. If $A \subset S$, the closure of A will be denoted A^*, the boundary of A by Bd (A). The empty set will be denoted $\square$. Cohomology is that of Alexander-Spanier [10] with coefficient group arbitrary. If $x \in S$, we use $\Gamma(x)$ to denote the closed subsemigroup $\{x, x^2, x^3, x^4, \cdots\}^*$ and E to denote $\{z: z = z^2\}$, the set of idempotents. If $e \in E$, then e is contained in a unique maximal subgroup which will be denoted $H(e)$.

If S is a compact semigroup, then S has a minimal ideal, $M(S)$, contained in all ideals of S. The minimal ideal of $\Gamma(x)$ is a single group.

A thread T is a semigroup on a closed, real arc such that one endpoint is a zero and the other an identity. If these are the only two idempotents, then T is isomorphic to either J_1 or J_2 and will be called a simple thread. If T is not simple, then $T \backslash E$ is the disjoint union of open intervals I_i, and each I_i^* is iseomorphic to either J_1 or J_2. Moreover, if T is given the cutpoint order with zero for the minimal element and x and y are not in the same interval, then $xy = yx = \min\{x, y\}$ [6]. A most important theorem on threads was proved by R. J. Koch [4] in 1964:

Theorem. *Let S be a compact, connected semigroup with identity u and minimal ideal $M(S) \neq \square$. If each subgroup of S is totally disconnected, then there is a thread which has u for an identity and meets the minimal ideal.*

Received October 1, 1968.