MOST TORI ARE EXTENDIBLE TO AN OPEN SET

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1. Introduction. If $A \subset C^n$, we say that A is extendible to a connected subset B of C^n if $A \subseteq B$, and if every function holomorphic about A is the restriction of a function holomorphic about B. When is a set A extendible to a B containing an open set? Theorem 1 below indicates the answer is "almost always".

Let $\mathfrak{M} = \{f\colon T^{n+1} \to C^n, f \text{ is } C^\infty\}$ where T^{n+1} is the $(n+1)$ torus, $\overbrace{S^1 \times \cdots \times S^1}^{n+1 \text{ times}}$. Give $\mathfrak{M}$ the C^k topology: uniform convergence of derivatives up to order k, with k sufficiently large ($k > n$).

Theorem 1. *There exists an open and dense subset $\mathcal{O}$ of $\mathfrak{M}$ such that $f \in \mathcal{O}$ implies that $f(T^{n+1})$ is extendible to a set containing an open subset of C^n.*

It is interesting to compare this "holomorphic hull" theorem with its "convex hull" analogue, whose proof is easy to obtain.

Let $\mathfrak{N} = \{f\colon I \to R^n, f \text{ is } C^\infty\}$, where $I = [0, 1]$, and give $\mathfrak{N}$ the C^k topology. If $A \subseteq R^n$, let ch A denote the convex hull of A.

Theorem 1′. *There is an open and dense subset $\mathcal{U}$ of $\mathfrak{N}$ such that $f \in \mathcal{U}$ implies that* ch $f(I)$ *contains an open set.*

The proof of Theorem 1′ depends upon being able to create a small 'bump' in a given $f\colon I \to R^n$. This is easily done by adjusting f to make $f', f'', \cdots, f^{(n)}$ linearly independent. Creation of an appropriate 'bump' in a given $f : T^{n+1} \to C^n$ to prove Theorem 1 is not as obvious—we must use the local criterion for extendibility developed in [2].

Note further that $n + 1$ is minimal for Theorem 1. If we consider $\mathfrak{M}' = \{f : T^n \to C^n, f \text{ is } C^\infty\}$ with a C^k topology, then the conclusion is no longer true. Indeed, for $n = 2$, R. O. Wells [4] shows that there are open sets S_1 and S_2 of $\mathfrak{M}'$ so that $f \in S_1$ implies $f(T^2)$ is not extendible, and if $f \in S_2$, then $f(T^2)$ is extendible to at least a three-dimensional subset of C^2.

Theorem 1 tends to support Bishop's remark [1]: "It is thought that a manifold $M^{n+1} \subset C^n$ has, in general, the property that holomorphic functions in a neighborhood of M extend to be holomorphic in some fixed open set." The reasoning presented here is purely local and does not depend on special properties of T^{n+1}. A detailed discussion of Bishop's statement, for arbitrary M^{n+1}, is contained in [3], where more precise information is obtained by using general transversality theorems. The proof of Theorem 1 in this paper contains some points essentially different from [3]. Also, the transversality is isolated and confined to three rather simple observations. So this proof is perhaps more palatable to the analyst.

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