AN INDICATOR DIAGRAM, II. THE L_p CONVOLUTION THEOREM FOR CONNECTED UNIMODULAR GROUPS

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1. **Introduction.** In this paper we continue to deal with a problem in harmonic analysis on locally compact unimodular groups posed in [3]. Suppose G is a locally compact unimodular group and $p \geq 1$. We say that the "L_p convolution theorem" is valid for G if: for every $f \in L_2(G)$ and $h \in L_p(G)$, we have $\int_G |f(x\xi^{-1})| |h(\xi)| \, d\xi < \infty$ a.e. and $||f * h||_2 \leq A_p \, ||f||_2 \, ||h||_p$. Here $(f * h)(x) = \int_G f(x\xi^{-1}) h(\xi) \, d\xi$, $x \in G$, and A_p is a constant that depends only on p.

The theorem is, of course, true for any G when $p = 1$. In addition, if G is compact, it is easily seen to be true for any $p \geq 1$. However if G is not compact, the theorem is false for $p \geq 2$ (see [3, §2] for $p > 2$ and [8] for $p = 2$). We are therefore primarily interested in categorizing those groups for which the Lp convolution theorem is true when $1 < p < 2$. Briefly, the results of this paper effect a reduction of this problem for connected unimodular groups to the case of connected semisimple Lie groups.

In order to state the exact theorem we need to make several definitions. First suppose $\mathcal{R}$ is a collection of topological groups. We say that a topological group G is a *compact extension of* $\mathcal{R}$ if there exists $H \subseteq G$, a compact normal subgroup, such that $G/H \in \mathcal{R}$. Next, let $\mathcal{S}$ denote the collection of connected semisimple Lie groups with finite center. For convenience in stating our theorem, we assume $\mathcal{S}$ contains the trivial group. Let $\mathcal{S}_0$ be the subcollection consisting of those members of $\mathcal{S}$ for which the L_p convolution theorem is true for all p, $1 < p < 2$. It follows from the above remarks and those in [4, §1] that $\mathcal{S}_0$ includes: all connected compact semisimple Lie groups, all the complex classical groups, and the Lorentz groups $SO_e(n, 1)$, $n \geq 2$. As a consequence of [3, Theorem 2 and Lemma 12], $\mathcal{S}_0$ also contains the groups obtained from these by (i) forming direct products, (ii) factoring out finite central subgroups, (iii) extending by finite central subgroups (i.e. $D \subseteq G$, finite central subgroup such that $G/D \in \mathcal{S}_0 \Rightarrow G \in \mathcal{S}_0$). It is conjectured that $\mathcal{S}_0$ equals $\mathcal{S}$ [3, §6], but that is still an open question. We shall prove: Let G be a connected unimodular group. Then the L_p convolution theorem is true for G for all p, $1 < p < 2$, if and only if G is a compact extension of $\mathcal{S}_0$.

Notation. We assume the reader is familiar with the results, conventions, and notation established in [3]. In particular, we shall use freely all the inclusion properties of the indicator diagram $\Delta(G)$ derived in [3]. Recall $C_0(G)$ = the continuous functions of compact support. We also denote by $C_P(G)$ the set of

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