SOME METRICAL THEOREMS IN NUMBER THEORY II.

By Walter Philipp

1. Introduction. Let $[a_1, a_2, \cdots]$ be the continued fraction expansion of $x \in (0, 1]$. The $a_n = a_n(x)$ can be viewed as random variables defined on an appropriate probability space. Let $\langle\varphi(n)\rangle$ be any sequence of positive real numbers. A classical theorem of Bernstein [12; 67] says that the event

$$a_n(x) \geq \varphi(n) \tag{1}$$

occurs infinitely often for almost no or almost all x according as the series $\sum 1/\varphi(n)$ converges or diverges. Of course, if the random variables $a_n(x)$ were independent, Bernstein's theorem would follow immediately from the Borel–Cantelli Lemma since it is not difficult to see that the Lebesgue measure λ of the set (1) multiplied by $\varphi(n)$ remains between two fixed positive constants c_1 and c_2, i.e. $0 < c_1 \leq \varphi(n) \cdot \lambda\{x: a_n(x) \geq \varphi(n)\} \leq c_2$. Similarly, the law of the iterated logarithm could be applied since the indicators of the sets (1) are uniformly bounded. However, as Chatterji [3] showed, any measure μ with the property that the $a_n(x)$ are independent with respect to μ is singular (with respect to λ). Thus this is not the right line of attack. (For an elegant proof of Chatterji's result see F. Schweiger [23].

In Chapter 2 it is shown that the $a_n(x)$ satisfy a certain mixing condition i.e. they are asymptotically independent. In earlier papers [15], [19] and [21] it was shown that the Borel–Cantelli Lemma, the law of the iterated logarithm and the central limit theorem continue to hold if the random variables satisfy certain mixing conditions rather than being independent. These theorems can be applied directly to the above situation and we obtain slight improvements over some results of Doeblin [5].

Theorem 1. *Let $A(N, x)$ be the number of integers $n \leq N$ satisfying* (1). *Let $\varphi(n) \to \infty$ be a sequence of integers with $\sum 1/\varphi(n) = \infty$. Set*

$$\Phi(N) = \frac{1}{\log 2} \sum_{n \leq N} \log\left(1 + \frac{1}{\varphi(n)}\right)$$

Then

$$\limsup_{N \to \infty} \frac{|A(N, x) - \Phi(N)|}{\sqrt{2\Phi(N) \log\log \Phi(N)}} = 1$$

almost everywhere.

A similar theorem holds if the $\varphi(n)$ are not assumed to tend towards ∞.

Received September 5, 1968. Research supported in part by the National Science Foundation Grant GP 7537.