CANCELLATIVE MEDIAL MEANS ARE ARITHMETIC

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A characterization of arithmetic mean on the real line was first given by Kolmogoroff [14] and Nagumo [17]. Introducing mediality, Aczel [2; 1; 281 ff.] later gave a very neat characterization and extended their results to the non-commutative case. Fuchs [11] showed Aczel's results held for arbitrary totally ordered spaces. An algebraic treatment of the notion of arithmetic mean has been given by Evans [10] and others.

This paper extends Aczel's results to the n-dimensional case, a particular instance of which is now stated.

Suppose that M is an open or closed n-cell and that $(x, y) \to xy$ is a continuous binary operation on M which is *idempotent* $(xx = x)$, *commutative* $(xy = yx)$, *medial* $(xy \cdot uv = xu \cdot yv)$, and cancellative. Then there exists a homeomorphism f of M onto a subset of R^n such that

$$f(xy) = \frac{f(x) + f(y)}{2} \qquad \text{for all} \quad x, y \quad \text{in} \quad M.$$

On the one hand, this proposition lies in the domain of "composite functional equations" [1], and on the other hand, it is a representation theorem for a very special non-associative topological algebra.

Our proof, however, is quite different from Aczel's, which relies heavily on the ordinal properties of the real line. Roughly put, we supply the machinery which enables us to draw sufficiently upon existing knowledge of topological semigroups to prove the result. The technique used, based on Theorem A, is of interest in its own right and may be useful in other contexts.

Although its connection with this paper is small, the problem of determining those spaces admitting a *mean* (continuous, commutative, idempotent binary operation) should be mentioned. The study of this problem was initiated by Aumann [3] and continued by Eckmann [7], and Eckmann, Ganea, and Hilton [8], the major tool of the latter authors being algebraic topology. More recently Philip Bacon has studied the problem [4; 5].

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1. Theorem A improves some results from [18] and forms the heart of our technique in the proof of Theorem B. To establish notation, we review briefly the construction of a special direct limit which we will use. We refer the reader to Spanier [19; 3] or Eilenberg and Steenrod [9; 212 ff.] for further details.

Let G be a topological groupoid and $f\colon G \to G$ be a homomorphism. We

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