COTOPOLOGY FOR METRIZABLE SPACES

By J. M. Aarts, J. de Groot and R. H. McDowell

1. Introduction. In [4] J. de Groot introduced the notion of "subcompactness", and proved a unifying Baire category theorem. The notion of cotopology grew out of this work. The basic idea is to associate with a topological space $(X, \mathcal{O})$ a new space $(X, {}^*\mathcal{O})$, where ${}^*\mathcal{O}$ is a coarser topology than $\mathcal{O}$, of a very special kind.

In §2 we define cotopologies and cocontinuity. In §3 we introduce the concept "cocompactness", and give several representative properties of cocompact spaces. Our main results appear in §4, where we prove (Theorem 1) that a metrizable space is topologically complete if and only if it is cocompact, and (Theorem 2) that a separable metrizable space is cocompact if and only if it is a cocontinuous image of the Cantor set.

The authors are of the opinion that cocompactness is the topological core of the completeness property for metric spaces. This is already clear from Theorem 1. From a topological point of view, cocompact spaces are a far better generalization of completely metrizable spaces than are spaces complete in some uniform structure. For example, the Baire category theorem holds in cocompact spaces. It is not known whether G_δ-subsets of compact Hausdorff spaces are cocompact.

Theorem 2 generalizes the well-known result that the compact metric spaces are precisely the continuous images of the Cantor set.

2. Cotopology. Cotopology may be roughly defined as that part of topology in which cospaces of a space X are used to study properties of X.

Fundamental definition. Let $T = (X, \mathcal{O})$ be a topological space. A topology ${}^*\mathcal{O}$ on X is called a cotopology of $\mathcal{O}$—and ${}^*T = (X, {}^*\mathcal{O})$ is called a *cospace* of T—if

(i) ${}^*\mathcal{O}$ is weaker than $\mathcal{O}$: ${}^*\mathcal{O} < \mathcal{O}$, and

(ii) for each point x and each closed neighborhood V of x in T there exists a neighborhood U of x in T such that U is contained in V and U is closed in ${}^*T$ (and hence in T).

For regular spaces, condition (ii) can be replaced by the following equivalent condition.

(iii) Each point x has a neighborhood base in T the elements of which are closed in ${}^*T$.

The identity map of X induces a one-to-one continuous map of T onto ${}^*T$ (because of (i)) of a special kind (because of (ii)) which we will call a *compression*

Received July 4, 1968.