SEQUENCES AND INVERSIONS

By L. Carlitz

1. Let $k_1, k_2, \cdots, k_n$ be fixed non-negative integers and put

$$m = k_1 + k_2 + \cdots + k_n . \tag{1.1}$$

We consider sequences of integers

$$\sigma = (a_1, a_2, \cdots, a_m), \tag{1.2}$$

where exactly k_1 of the a_j equal 1, exactly k_2 equal 2, $\cdots$, exactly k_n equal n. An *inversion* is a pair i, j such that

$$i < j, \qquad a_i > a_j . \tag{1.3}$$

We seek

$$N_n(r) = N(r; k_1, k_2, \cdots, k_n), \tag{1.4}$$

the number of sequences (1.2) with exactly r inversions. In the special case $k_1 = k_2 = \cdots = k_n = 1$ the problem has been discussed by Netto [1; 94]. Also it is known that when $k_1 = k_2 = \cdots = k_n = 1$,

$$\sum_{r=0}^{\frac{1}{2}n(n-1)} N_n(r)x^r = \frac{(1-x)(1-x^2)\cdots(1-x^n)}{(1-x)^n}. \tag{1.5}$$

This result was stated as a problem in Elemente der Mathematik [2].

C. A. Church has informed the writer that the formula (1.5) was first obtained by Rodrigues [3].

In the present note we show, that for arbitrary nonnegative $k_1, k_2, \cdots, k_n$

$$\sum_{r=0} N(r; k_1, k_2, \cdots, k_n)x^r = \frac{[k_1 + k_2 + \cdots + k_n]!}{[k_1]!\,[k_2]! \cdots [k_n]!}, \tag{1.6}$$

where

$$[k]! = (1-x)(1-x^2)\cdots(1-x^n).$$

It is clear that (1.1) reduces to (1.5) when $k_1 = k_2 = \cdots = k_n = 1$. Some additional properties of $N(r; k_1, \cdots, k_n)$ are also obtained.

The quotient

$$[k_1, k_2, \cdots, k_n] = \frac{[k_1 + k_2 + \cdots + k_n]!}{[k_1]!\,[k_2]! \cdots [k_n]!} \tag{1.7}$$

evidently reduces to the multinomial coefficient

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