A NOTE ON QUASITRIANGULAR OPERATORS

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1. Introduction. In this note $\mathcal{H}$ will denote a fixed separable, infinite-dimensional, complex Hilbert space, and the algebra of all bounded linear operators on $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be quasitriangular if there exists a sequence $\{P_n\}$ of finite rank (orthogonal) projections on $\mathcal{H}$ converging strongly to 1 such that

$$(*) \qquad \|P_nTP_n - TP_n\| \to 0.$$

In [7], Halmos initiated the study of quasitriangular operators, and he proved there that an equivalent definition of quasitriangularity may be given as follows. Let $\mathcal{P}$ denote the directed set of all finite rank projections in $\mathcal{L}(\mathcal{H})$ under the usual ordering ($P \leq Q$ if and only if $(Px, x) \leq (Qx, x)$ for all x in $\mathcal{H}$). For a fixed T in $\mathcal{L}(\mathcal{H})$, the map $P \to \|PTP - TP\|$ is a net on $\mathcal{P}$, and T is quasitriangular if and only if

$$(**) \qquad \liminf_{P \in \mathcal{P}} \|PTP - TP\| = 0.$$

Halmos also proved that the direct sum of countably many quasitriangular operators is quasitriangular [7, Theorem 4], and he asked whether for every $T \in \mathcal{L}(\mathcal{H})$, either T or T^* is quasitriangular [7, Question 3].

In this note, we show that the answer to this question is no by exhibiting a large class of operators T such that neither T nor T^* is quasitriangular. We also generalize the above-mentioned [7, Theorem 4] by proving that every upper-triangular matrix (with operator entries) whose diagonal entries are quasitriangular is also quasitriangular. This enables us to prove that all operators with finite spectrum and all polynomially compact operators are quasitriangular. Finally, we give two additional equivalent definitions of quasitriangularity; these enable us to show that if A is quasitriangular, then so is every operator similar to A, and that if A is not quasitriangular, then neither is the direct sum of A with any zero operator.

2. Some non-quasitriangular operators. The following lemma, which is essentially [7, Theorem 3], is a basic tool for constructing non-quasitriangular operators.

Lemma 2.1. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is bounded below and T^* has a non-trivial null space. Then T is not quasitriangular.*

Proof. Let $\alpha > 0$ be such that $\|Tx\| \geq \alpha \|x\|$ for all x in $\mathcal{H}$, and let x_0 be a non-zero vector in the null space of T^*. Let P_0 be the rank-one projection

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