EXTREME PROPERTIES OF PRODUCTS OF QUADRATIC FORMS

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For a positive linear transformation A on E_n, a finite dimensional unitary space, the minimum of a product of quadratic forms of A is given in [2]. The maximum is a special case of a theorem of M. Marcus and J. L. McGregor [6]. In this article we study certain generalizations of these ideas.

1. Definitions and notations. The inner product of two vectors α and β will be denoted by (α, β). The determinant of a linear transformation A on E_n will be denoted by $\det A$. A Hermitian linear transformation is called positive if and only if $(A\xi, \xi) > 0$ for all $\xi \neq 0$. An orthonormal set $\{\alpha_1, \cdots, \alpha_k\}$ will be indicated by $\{\alpha_p\}$ o.n. A subspace spanned by the set $\{\gamma_1, \cdots, \gamma_p\}$ will be denoted by $[\gamma_1, \cdots, \gamma_p]$. The expression $A \mid M$ denotes the linear transformation A restricted to the subspace M, as defined in [1].

2. Theorem. *Let A be a positive linear transformation on E_n with proper values $m_1 \geq \cdots \geq m_n$. Then*

$$\sup_{\{\xi_p\}\text{o.n.}} F_r((A\xi_1, \xi_1), \cdots, (A\xi_k, \xi_k)) = \binom{k}{r}\left(\frac{1}{k}\sum_{j=1}^{k} m_j\right)^r,$$

where F_r denotes the r-th elementary symmetric function, i.e.,

$$F_r(t_1, \cdots, t_k) = \sum_{1 \leq i_1 < \cdots < i_r < k} t_{i_1} \cdot \cdots \cdot t_{i_r}.$$

The proof is due to M. Marcus and J. L. McGregor [6].

3. Theorem. Let A be a positive linear transformation on E_n with proper values $m_1 \geq \cdots \geq m_n$. Then

$$\inf_{\substack{M \\ \dim M = h}} \sup_{\substack{\{\xi_p\}\text{o.n.} \\ \xi_p \varepsilon M}} F_r((A\xi_1, \xi_1), \cdots, (A\xi_k, \xi_k)) = \binom{k}{r}\left(\frac{m_{n-h+1} + \cdots + m_{n-h+k}}{k}\right)^r,$$

where $1 \leq r \leq k \leq h \leq n$, and F_r is the same as in 2.

Proof. Let M be any subspace of E_n such that $\dim M = h$ and let P be the orthogonal projection on M. Then, for $\xi \varepsilon M$, it follows that $(A \mid M)\xi = PA\xi$. Let $s_1 \geq \cdots \geq s_h$ be the proper values of $A \mid M$. Then, by 2, we have

$$\sup_{\substack{\{\xi_p\}\text{o.n.} \\ \xi_p \varepsilon M}} F_r((A\xi_1, \xi_1), \cdots, (A\xi_k, \xi_k)) = \binom{k}{r}\left(\frac{1}{k}\sum_{j=1}^{k} s_j\right)^r.$$

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