COMPLETION REGULARITY OF (WEAKLY) BOREL MEASURES

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Let X be a locally compact, Hausdorff topological space. We shall use $\mathcal{B}$, $\mathcal{B}_\delta$, $\mathcal{B}_w$, and $\mathcal{B}_{\delta w}$ to denote the σ-rings generated by the compact, compact G_δ, closed, and closed G_δ subsets of X, respectively. Following [1], we shall call them the classes of *Borel, Baire, weakly Borel* (w.B.), and *weakly Baire* (w.Ba.) subsets of X, respectively. Of course, $\mathcal{B}_\delta \subset \mathcal{B} \subset \mathcal{B}_w$ and $\mathcal{B}_\delta \subset \mathcal{B}_{\delta w} \subset \mathcal{B}_w$. *Measures defined on these classes of sets which are finite on compact sets will carry the same names.*

The concept of a completion regular Borel measure is well known [3; 230]. In this paper we study an extension of this idea to w.B. measures, calling it by the same name "completion regular." We characterize those left [resp., right] invariant w.B. measures on a locally compact, Hausdorff topological group X which are completion regular (Corollary 4.5); a special case of this is the well-known result that every left [resp., right] invariant Borel measure on X is completion regular [3, 64.I]. In §3 we decompose a w.B. measure ν on a locally compact, Hausdorff topological space into the sum of two w.B. measures, one the "largest completion regular part of ν" and the other having zero "largest completion regular part" (see Theorem 3.1 and its corollaries). The latter will be called an anti-completion regular w.B. measure; a study of it, especially in relation to completion regular w.B. measures, is pursued in §2.

Finally, everything in this paper is valid if the class of w.Ba. subsets of X is instead defined to be the σ-algebra $\mathcal{B}_0$ generated by the zero-sets in X—i.e., if $\mathcal{B}_{\delta w}$ is replaced by $\mathcal{B}_0$ throughout. (A *zero-set* in X is a set of the form $f^{-1}(0)$ for some continuous real-valued function f on X.) The relevant facts are that $\mathcal{B}_\delta \subset \mathcal{B}_0 \subset \mathcal{B}_{\delta w}$ (if X is locally compact, Hausdorff) and $\mathcal{B}_0 = \mathcal{B}_{\delta w}$ if X is normal.

1. Preliminaries.

We use extensively the results and methods of [7]. We shall list some of the more basic ones here and give specific references for the remaining ones as we use them.

Let X be simply a set, $\mathcal{S}$ be a σ-ring of subsets of X, and μ, ν be measures on $\mathcal{S}$. ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $E \in \mathcal{S}$ and $\mu(E) = 0$ [3; 124]. Following [5], we say that ν is *S-singular* with respect to μ, denoted $\nu \mathrm{S} \mu$, if for each $E \in \mathcal{S}$ there corresponds a set $F \in \mathcal{S}$ such that $F \subset E$, $\nu(F) = \nu(E)$, and $\mu(E) = 0$. A set $A \subset X$ is said to be *locally measurable* if $A \cap E \in \mathcal{S}$ for every $E \in \mathcal{S}$ [1; 35]. As is customary [3; 126], we say that ν is *singular* with respect to μ, denoted $\nu \perp \mu$, if there is a locally measurable set A such that $\nu(E \cap A) = 0 = \mu(E - A)$ for every $E \in \mathcal{S}$.

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