SOME RESULTS ON PSEUDO VALUATIONS

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1. Introduction and preliminary results. In this paper all rings are commutative, associative, and have identity. If A and B are two rings, B an overring of A, assume that A and B have the same identity. The symbol R_∞ will denote the extended real number system.

A *pseudo valuation* on a ring A is a mapping $w: A \to R_\infty$ with the properties:

(i) $w(0) = \infty, w(1) = 0, w(x) \geq 0,$
(ii) $w(x - y) \geq \min \{w(x), w(y)\},$
(iii) $w(xy) \geq w(x) + w(y)$ for all $x, y \in A$.

w is called a *homogeneous pseudo valuation* in case:
(iv) $w(x^n) = n \cdot w(x)$ for all positive integers n and for all $x \in A$.

w is a *valuation* on A in case:
(v) $w(xy) = w(x) + w(y)$ for all $x, y \in A$.

Suppose that A is a ring and $\mathfrak{a}$ is an ideal of A. Consider the non-negative integral powers of $\mathfrak{a}$, $\{\mathfrak{a}^n\}$. Define a mapping $v_\mathfrak{a} : A \to R_\infty$ such that $v_\mathfrak{a}(x) = n$ if $x \in \mathfrak{a}^n$, $x \notin \mathfrak{a}^{n+1}$ and $v_\mathfrak{a}(x) = \infty$ if $x \in \mathfrak{a}^n$ for all n. The map $v_\mathfrak{a}$ is clearly seen to be a pseudo valuation on A. Call $v_\mathfrak{a}$ the *pseudo valuation associated with the ideal* $\mathfrak{a}$. D. Rees studied this type of pseudo valuation in [3], [4], [5]. His results and definitions which will be needed for this paper will be summarized here. Suppose w is a pseudo valuation on A. Define $\bar{w}(x) = \lim_{n=1}^{\infty} w(x^n)/n$ for each $x \in A$. This limit exists for every $x \in A$ and is actually equal to $\sup \{w(x^n)/n: n = 1, 2, \cdots \}$. Also $\bar{w}$ is a homogeneous pseudo valuation on A. Hence, every pseudo valuation on A may be homogenized.

Now suppose that $v_1, \cdots, v_k$ are valuations on A. Define $w(x) = \min \{v_1(x), \cdots, v_k(x)\}$ on A. Then w is called a *subvaluation.* (i.e., A subvaluation is a function that can be written as a minimum of valuations.) It is clear that w is a homogeneous pseudo valuation. The set $\{v_1, \cdots, v_n\}$ is a *representation* of w. The representation is *irredundant* in case for each i, there exists an $x \in A$ such that $w(x) = v_i(x)$, but $w(x) < v_j(x)$ for every $j \neq i$. Let $Aw = \{x \in A: w(x) \neq \infty\}$. A subset S of Aw is said to be *w-consistent* if for any finite set of elements $a_1, a_2, \cdots, a_n \in S$, $w(a_1a_2 \cdots a_n) = w(a_1) + w(a_2) + \cdots + w(a_n)$. For each w-consistent subset S there exists a maximal w-consistent subset S' such that $S' \supset S$. If $w = \min \{v_1, \cdots, v_k\}$ is irredundant, then there exist exactly k maximal w-consistent subsets

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